Entire and meromorphic functions
Problem set 10 (due Tuesday, January 30)

Problems 1–3 deal with the entire function

$$f(z) = \int_0^z \exp(-t^2)dt.$$

1. Let $\delta > 0$. Show that, as $|z| \to \infty$,

$$f(z) = -(1 + o(1))\frac{\exp(-z^2)}{2z} \quad \text{for } \left|\arg z \pm \frac{\pi}{2}\right| \leq \frac{\pi}{4} - \delta,$$

$$f(z) - \frac{\sqrt{\pi}}{2} = -(1 + o(1))\frac{\exp(-z^2)}{2z} \quad \text{for } |\arg z| \leq \frac{\pi}{4} - \delta$$

and

$$f(z) + \frac{\sqrt{\pi}}{2} = -(1 + o(1))\frac{\exp(-z^2)}{2z} \quad \text{for } |\arg z - \pi| \leq \frac{\pi}{4} - \delta.$$

Here the $o(1)$-terms are uniform in the given ranges for θ.

Hint. Use integration by parts. Note also that

$$\int_0^\infty \exp(-t^2)dt = \frac{\sqrt{\pi}}{2}.$$

2. Show that

$$T(r, f) = (1 + o(1))\frac{r^2}{\pi} \quad \text{and} \quad m\left(r, \frac{1}{f \pm \frac{\sqrt{\pi}}{2}}\right) \geq (1 - o(1))\frac{r^2}{2\pi}$$

as $r \to \infty$.

3. Show that

$$\delta\left(\pm\frac{\sqrt{\pi}}{2}, f\right) = \frac{1}{2}.$$

Hint. Use the previous problem and the deficiency relation.