Prof. Dr. W. Bergweiler Winter semester 2017/18

Entire and meromorphic functions
Problem set 9 (due Tuesday, January 23)

Let $\tau \in \mathbb{C}$ with $\operatorname{Im} \tau > 0$ and $\Lambda = \{m + n\tau \colon m, n \in \mathbb{Z}\}$ be as in Problem set 8 (and Problem 3 of Problem set 6).
A meromorphic function $f \colon \mathbb{C} \to \widehat{\mathbb{C}}$ satisfying $f(z+1) = f(z)$ and $f(z + i\tau) = f(z)$ for all $z \in \mathbb{C}$ and hence $f(z + \omega) = f(z)$ for all $\omega \in \Lambda$ and all $z \in \mathbb{C}$ is called doubly-periodic or elliptic. A set of the form $\{z_0 + s + t\tau \colon 0 \leq s < 1,\ 0 \leq t < 1\}$ with $z_0 \in \mathbb{C}$ is then called a period parallelogram (for f).

1. Let f be an elliptic function and P a period parallelogram. Let $p_1, \ldots, p_n$ be the poles of f in P. Show that
$$\sum_{j=1}^{n} \operatorname{res}(f, p_j) = 0.$$
Here $\operatorname{res}(f, p_j)$ denotes the residue of f at p_j.

2. Let f be an elliptic function, P a period parallelogram and $a \in \mathbb{C}$. Show that, counting multiplicities, the number of a-points of f in P is equal to the number of poles of f in P.

   Remark. The number of poles in a period parallelogram is also called the order of f. However, it should not be confused with the order of growth $\rho(f)$.

3. Let $\wp$ be the Weierstraß $\wp$-function corresponding to Λ. Show that
$$\wp'(1/2) = \wp'(\tau/2) = \wp'((1+\tau)/2) = 0.$$
Put
$$e_1 = \wp(1/2), \quad e_2 = \wp((1+\tau)/2) \quad \text{and} \quad e_3 = \wp(\tau/2).$$
Use Problem 2 to show that $e_j \neq e_k$ for $j \neq k$.
Show also that
$$\wp'(z)^2 = 4(\wp(z) - e_1)(\wp(z) - e_2)(\wp(z) - e_3).$$
Compare Problem 4 of Problem set 8.

4. Show that $\wp$ has four totally ramified values.

Remark. In the theory of elliptic functions one usually considers meromorphic functions f satisfying $f(z + \omega_1) = f(z + \omega_2) = f(z)$ where $\tau := \operatorname{Im}(\omega_2/\omega_1)$ satisfies $\operatorname{Im} \tau > 1$, and correspondingly $\Lambda = \{m\omega_1 + n\omega_2 \colon m, n \in \mathbb{Z}\}$. We have restricted to the special case $\omega_1 = 1$, but all results given extend to the general case in an obvious way. In fact, the general case can be reduced to the special case by considering $f(z\omega_1)$ instead of $f(z)$.