Entire and meromorphic functions
Problem set 8 (due Tuesday, January 16)

1. Let $\tau \in \mathbb{C}$ with $\operatorname{Im} \tau > 0$ and $\Lambda = \{m + n\tau \colon m, n \in \mathbb{Z}\}$ be as in Problem 3 of Problem set 6. Let

$$\sigma(z) = z \prod_{\omega \in \Lambda \setminus \{0\}} E\Big(\frac{z}{\omega}, 2\Big)$$

be the canonical product with zeros at the points of Λ and put $\zeta(z) = \sigma'(z)/\sigma(z)$ and $\wp(z) = -\zeta'(z)$.

Use Problem 3 of Problem set 6 to show that

$$\zeta(z) = \frac{1}{z} + \sum_{\omega \in \Lambda \setminus \{0\}} \left(\frac{1}{z - \omega} + \frac{1}{\omega} + \frac{z}{\omega^2} \right)$$

and hence

$$\wp(z) = \frac{1}{z^2} + \sum_{\omega \in \Lambda \setminus \{0\}} \left(\frac{1}{(z - \omega)^2} - \frac{1}{\omega^2} \right)$$

and

$$\wp'(z) = -\sum_{\omega \in \Lambda} \frac{2}{(z - \omega)^3}.$$

Deduce that $\wp'(z + \omega) = \wp'(z)$ for all $\omega \in \Lambda$ and all $z \in \mathbb{C}$.

Show that $\wp$ is an even function, i.e., $\wp(z) = \wp(-z)$ for all $z \in \mathbb{C}$, and use this to show that also $\wp(z + \omega) = \wp(z)$ for all $\omega \in \Lambda$ and all $z \in \mathbb{C}$.

Remark. The functions σ, ζ and $\wp$ are called the Weierstraß sigma-, zeta- and p-function.

2. Show that the Laurent series expansion of ζ and $\wp$ around 0 are given by

$$\zeta(z) = \frac{1}{z} - \sum_{n=2}^{\infty} G_n z^{2n-1} \quad \text{and} \quad \wp(z) = \frac{1}{z^2} + \sum_{n=2}^{\infty} (2n - 1) G_n z^{2n-2},$$

with

$$G_n = \sum_{\omega \in \Lambda \setminus \{0\}} \frac{1}{\omega^{2n}}.$$

3. Let $\tau \in \mathbb{C}$ with $\operatorname{Im} \tau > 0$ and $\Lambda = \{m + n\tau \colon m, n \in \mathbb{Z}\}$ be as before. Let f be an entire function satisfying $f(z + \omega) = f(z)$ for all $\omega \in \Lambda$ and all $z \in \mathbb{C}$. Show that f is constant.

4. Use the two previous problems to show that $\wp$ satisfies the differential equation

$$\wp'(z)^2 = 4\wp(z)^3 - g_2 \wp(z) - g_3$$

where $g_2 = 60 G_2$ and $g_3 = 140 G_3$.