Entire and meromorphic functions
Problem set 7 (due Monday, January 8)

1. Determine the product representation of $\cos \pi z$ according to the Hadamard factorization theorem.

2. Show that
$$\pi \cot \pi z = \frac{1}{z} + \sum_{n\in\mathbb{Z}\setminus\{0\}} \left(\frac{1}{z-n} + \frac{1}{n} \right) = \frac{1}{z} + \sum_{n=1}^{\infty} \frac{2z}{z^2-n^2}$$
and
$$\frac{\pi^2}{\sin^2 \pi z} = \sum_{n=-\infty}^{\infty} \frac{1}{(z-n)^2}$$
for $z \in \mathbb{C}\setminus\mathbb{Z}$.

Hint. Use Problem 2 of Problem Set 6.

3. Show that
$$\Gamma(z)\Gamma(1-z) = \frac{\pi}{\sin \pi z},$$
$$\Gamma(\tfrac{1}{2}) = \sqrt{\pi}$$
and
$$\Gamma(z)\Gamma(z+\tfrac{1}{2}) = 2^{1-2z}\sqrt{\pi}\,\Gamma(2z).$$

4. Show that if $\operatorname{Re} z > 0$, then
$$\Gamma(z) = \int_0^\infty e^{-t} t^{z-1} dt.$$

Hint. Let
$$P(z,n) := \frac{n^z}{z} \prod_{j=1}^{n} \frac{j}{j+z} = n^z \frac{1\cdot 2 \cdot \ldots \cdot n}{z\cdot(z+1)\cdot\ldots\cdot(z+n)}.$$
Show that
$$\Gamma(z) = \lim_{n\to\infty} P(z,n)$$
as well as
$$\int_0^n \left(1-\frac{t}{n}\right)^n t^{z-1} dt = n^z \int_0^1 (1-u)^n u^{z-1} du = P(z,n).$$