Prof. Dr. W. Bergweiler Winter semester 2017/18

Entire and meromorphic functions
Problem set 5 (due Friday, December 1)

1. Let f be an entire function and $0 < r < R$. Show that

$$M(r,f) \leq rM(r,f') + |f(0)|$$

and

$$M(r,f') \leq \frac{M(R,f)}{R-r}.$$

Use this to prove that $\rho(f') = \rho(f)$ and $\lambda(f') = \lambda(f)$.

Show also that given $\alpha > 1$ there exists a subset E of $(0,\infty)$ of finite measure such that

$$M(r,f') \leq M(r,f)(\log M(r,f))^\alpha \quad \text{for } r \notin E.$$

Show by an example that this fails for $\alpha = 1$.

2. Prove Theorem 7.3 of the lecture; that is, prove that if f is entire with Taylor series expansion

$$f(z) = \sum_{n=0}^{\infty} c_n z^n,$$

then

$$\rho(f) = \limsup_{n\to\infty} \frac{n \log n}{-\log |c_n|}.$$

Hint. Use the inequalities

$$|c_n| r^n \leq M(r,f) \leq \sum_{n=0}^{\infty} |c_n| r^n.$$

For one direction, assume $\rho(f) < \mu$ and choose r depending on n such that the estimate

$$|c_n| \leq \frac{M(r,f)}{r^n} \leq \frac{e^{r^\mu}}{r^n}$$

becomes "optimal".

For the other direction, assume that

$$\frac{n \log n}{-\log |c_n|} \leq \mu$$

and find N depending on r such that $|c_n| r^n \leq 2^{-n}$ for $n \geq N$ so that $M(r,f) \leq \sum_{n=0}^{N} |c_n| r^n + 1$.