Entire and meromorphic functions
Problem set 4 (due Friday, November 24)

1. Let $f(z) = e^{e^z}$. Show that there exists $c > 0$ such that
$$T(r, f) \leq c\frac{e^r}{\sqrt{r}}$$
for all $r > 0$.

Remark. We actually have
$$T(r, f) \sim \frac{e^r}{\sqrt{2\pi^3 r}}$$
as $r \to \infty$. If you are more ambitious, you can also show that.

2. Let f be meromorphic with $f(0) \neq 0$ and let $r > 0$. Show that
$$\frac{1}{2\pi}\int_0^{2\pi} m\left(r, \frac{1}{f - e^{i\varphi}}\right) d\varphi \leq \log 2.$$

3. Let $r > 0$ and $g\colon \partial D(0, r) \to \mathbb{R}$ continuous.

(a) Show that the function $f\colon D(0, r) \to \mathbb{C}$,
$$f(z) = \frac{1}{2\pi i}\int_{|z|=r} g(\zeta)\frac{\zeta + z}{\zeta - z}\frac{d\zeta}{\zeta} = \frac{1}{2\pi}\int_0^{2\pi} g(re^{i\theta})\frac{re^{i\theta} + z}{re^{i\theta} - z}d\theta$$
is holomorphic.

(b) Show that the function $u\colon D(0, r) \to \mathbb{R}$
$$u(z) = \frac{1}{2\pi}\int_0^{2\pi} g(re^{i\theta})K(z, r, \theta)d\theta,$$
with
$$K(z, r, \theta) = \frac{r^2 - |z|^2}{|re^{i\theta} - z|^2}.$$
as in Theorem 6.1, is the real part of the function f from part (a) and thus harmonic.

(c) Show that the function $h\colon \overline{D}(0, r) \to \mathbb{R}$,
$$h(z) = \begin{cases} u(z) & \text{if } |z| < r, \\ g(z) & \text{if } |z| = r, \end{cases}$$
is continuous.

4. Let $r > 0$ and $u\colon D(0, r) \to \mathbb{R}$ be harmonic. Show that there exists a holomorphic function $f\colon D(0, r) \to \mathbb{C}$ such that $u = \operatorname{Re} f$.