Entire and meromorphic functions
Problem set 2 (due Friday, November 10)

1. Prove Lemma 3.1 of the lecture.

2. Show that
$$\frac{1}{2\pi}\int_0^{2\pi}\log|e^{i\theta}-1|d\theta = 0.$$
Use this to complete the proof of Jensen's formula (Theorem 3.2).
Hint: Consider
$$\frac{1}{2\pi}\int_0^{2\pi}\log|re^{i\theta}-1|d\theta$$
for $0<r<1$ and take the limit $r\to 1$.

3. Let f be meromorphic in $\overline{D}(0,r)$. Suppose that $f(z)\neq 0,\infty$ for $|z|=r$ and let M and N denote the number of zeros and poles of f in $D(0,r)$, respectively. The argument principle says that
$$\frac{1}{2\pi i}\int_{|z|=r}\frac{f'(z)}{f(z)}dz = M-N.$$
   (a) Deduce the argument principle from the residue theorem. (This is usually done in a complex function theory course, so you probably just have to recall what was done there.)
   (b) Deduce the argument principle from Jensen's formula.

   Hint for (b): Show that if $w=u+iv$ is holomorphic, then $ru_r(z)-iu_\theta(z)=w'(z)z$; that is,
$$ru_r(re^{i\theta})-iu_\theta(re^{i\theta}) = w'(re^{i\theta})re^{i\theta}.$$
   Here u_r and u_θ denote the partial derivatives with respect to polar coordinates. More precisely, if $u\colon D\to\mathbb{R}$ is differentiable, $\tilde{D}=\{(r,\theta)\in(0,\infty)\times\mathbb{R}\colon re^{i\theta}\in D\}$ and $\tilde{u}\colon\tilde{D}\to\mathbb{C}$, $\tilde{u}(r,\theta)=u(re^{i\theta})$, then u_r is the partial derivative of $\tilde{u}$ with respect to the first variable and u_θ is the partial derivative of $\tilde{u}$ with respect to the second variable. So
$$u_r=\frac{\partial\tilde{u}}{\partial r}\quad\text{and}\quad u_\theta=\frac{\partial\tilde{u}}{\partial\theta}.$$
   Differentiate Jensen's formula with respect to r and use the above for $u=\log|f|$.