Entire and meromorphic functions
Problem set 1 (due Friday, November 3)

Definition: A Möbius transformation is a rational function of degree 1. A Möbius transform of the form $z \mapsto z + c$ with $c \in \mathbb{C}$ is called a *translation*, one of the form $z \mapsto cz$ with $c \in \mathbb{C}$, $|c| = 1$, is called a *rotation* and one of the form $z \mapsto cz$ with $c \in \mathbb{R}$, $c > 0$, is called a *dilation*. The Möbius transformation given by $z \mapsto 1/z$ is called the *inversion*.

1. Show that every Möbius transformation can be written as the composition of translations, rotations, dilations and inversions.

2. Let C be the set of all circles in $\mathbb{C}$ and let L be the set of all straight lines in $\mathbb{C}$, with the point ∞ added to each such straight line. Show that if T is a Möbius transformation, then $T(C \cup L) = C \cup L$.

   Hint: Use Problem 1.

3. Let $a_1, a_2, a_3, b_1, b_2, b_3 \in \widehat{\mathbb{C}} = \mathbb{C} \cup \infty$, with $a_j \neq a_k$ and $b_j \neq b_k$ for $j \neq k$. Show that there exists a Möbius transformation T such that $T(a_j) = b_j$ for $j \in \{1, 2, 3\}$.

   Hint: Reduce the general case to the special case $b_1 = 0$, $b_2 = 1$ and $b_3 = \infty$.

4. Let $f(z) = \cos z$ and $a \in \mathbb{C}$. For $r > 0$ let $n(r, a)$ denote the number of a-points of f in $\{z \in \mathbb{C} \colon |z| \leq r\}$. Show that $n(r, a) \sim 2r/\pi$ as $r \to \infty$.