Complex dynamics
Problem set 9 (due Tuesday, June 21)

1. Show that the infinite product
$$\prod_{k=1}^{\infty} \cos\left(\frac{z}{2^k}\right)$$
converges for all $z \in \mathbb{C}$.
Use the addition theorems of the trigonometric functions to find an expression for the partial products
$$\prod_{k=1}^{n} \cos\left(\frac{z}{2^k}\right)$$
and use this to determine the value of the infinite product.

2. For $p \in \{0, 1, 2, \dots\}$ and $z \in \mathbb{C}$ put
$$E_p(z) = (1-z)\exp\left(\sum_{j=1}^{p} \frac{z^j}{j}\right).$$
Here, as usual, $\sum_{j=1}^{0} \cdots = 0$ so that $E_0(z) = 1 - z$.
Show that
$$1 - E_p(z) = \frac{1}{p+1} z^{p+1} + O\big(z^{p+2}\big)$$
as $z \to 0$.
Hint. Consider $E_p'(z)$.

3. Let (z_k) be a sequence in $\mathbb{C}\backslash\{0\}$ such that $|z_k| \to \infty$ as $k \to \infty$. Let (p_k) be sequence of non-negative integers such that
$$\sum_{k=1}^{\infty} \left(\frac{R}{|z_k|}\right)^{p_k+1} < \infty$$
for all $R > 0$. Show that the infinite product
$$\prod_{k=1}^{\infty} E_{p_k}\left(\frac{z}{z_k}\right)$$
converges locally uniformly in $\mathbb{C}$.
Show also that the zeros of the entire function f defined by
$$f(z) = \prod_{k=1}^{\infty} E_{p_k}\left(\frac{z}{z_k}\right)$$
are precisely the z_k. Finally, show that the product always converges for the choice $p_k = k$.