Complex dynamics
Problem set 6 (due Tuesday, May 31)

1. Let $N \in \mathbb{N}$, $N \geq 2$, and $0 < \delta < 1/N$. Let $C_{N,\delta}$ be the set obtained in the same way as the Cantor middle third set, except that in each step from every interval I one removes $N-1$ equally spaced open intervals of the same length so that N equally spaced closed intervals remain, each of which has length δ times the length of I.

   Compute the Hausdorff dimension of $C_{N,\delta}$.

2. Let $\alpha > 1$ and let C_α be the set obtained in the same way as the Cantor middle third set, where in the k-th step from every interval I one removes $k-1$ equally spaced open intervals of the same length so that k equally spaced closed intervals remain, each of which has length $1/k^\alpha$ times the length of I.

   Compute the Hausdorff dimension of C_α.

3. The Sierpinski triangle, Sierpinski carpet and Menger sponge are higher dimensional analogues of the Cantor middle third sets.

   For the Sierpinski triangle one starts with a closed equilateral triangle and divides it into four triangles by connecting the midpoints of the sides. Then the open triangle in the middle is removed so that three closed triangles remain.

   For the Sierpinski carpet one starts with a closed square and divides it into nine squares (in the obvious way). Then the open square in the middle is removed so that eight closed squares remain.

   For the Menger sponge a closed cube is divided into 27 smaller cubes (in the obvious way). Now 7 of these 27 smaller cubes are removed, namely the one in the middle and the 6 smaller cubes corresponding to the midpoints of the 6 faces of the original cube. So 20 closed cubes remain.

   In each case, the process is continued inductively as in the case of the Cantor set.

   Compute the Hausdorff dimension of the Sierpinski triangle, Sierpinski carpet and Menger sponge.

   Additional questions: What happens if, for the Sierpinski triangle, we do not remove the triangle in the center, but some other triangle? The analogous question can also be asked for the other objects. What happens if we remove not one square in the construction of the Sierpinski carpet, but N squares, for some $N < 9$?. Similarly, what happens if we remove N cubes, where $N < 27$, in the construction of the Menger sponge?