Prof. Dr. W. Bergweiler Summer semester 2016

Complex dynamics
Problem set 5 (due Tuesday, May 24)

Recall that the postsingular set $P(f)$ of an entire function f is defined by

$$P(f) = \overline{\bigcup_{n=0}^{\infty} f^n(\operatorname{sing}(f^{-1}))}.$$

1. Let f be an entire function, U a component of $F(f)$ and $a \in \mathbb{C}$. Suppose that $f^{n_k}|_U \to a$ for some sequence (n_k) tending to ∞. Show that $a \in P(f)$.

   Hint. Use (and verify) the result from part 1 that if V is a domain and $\mathcal{F}$ is a family consisting of branches of f^{-1} defined in V, then $\mathcal{F}$ is normal.

2. Let f be an entire function for which $\operatorname{sing}(f^{-1})$ is finite. Suppose that all points of $\operatorname{sing}(f^{-1})$ are eventually mapped to repelling periodic cycles; that is, for all $w \in \operatorname{sing}(f^{-1})$ there exists n such that $f^n(w)$ is a repelling periodic point.

   Show that $J(f) = \mathbb{C}$.

   Show also that the hypotheses are satisfied for $f(z) = 2\pi i e^z$.

   Definition. A map $f\colon A \to B$, with $A, B \subset \mathbb{R}^d$, is called Lipschitz continuous if there exists a constant L such that $\|f(x) - f(y)\| \leq L\|x - y\|$ for all $x, y \in A$. The map f is called a bi-Lipschitz map if f is bijective and both f and f^{-1} are Lipschitz continuous.

   A map $f\colon A \to \mathbb{R}^d$ is called Hölder continuous with exponent $\alpha \in (0, 1]$, if there exists a constant C such that $\|f(x) - f(y)\| \leq C\|x - y\|^\alpha$ for all $x, y \in A$. So f is Lipschitz continuous if it is Hölder continuous with exponent 1.

3. Let $f\colon A \to \mathbb{R}^d$ be Hölder continuous with exponent α. Estimate $\dim f(A)$ in terms of $\dim A$ and α.

   Use this result to show that if $A, B \subset \mathbb{R}^d$ and $f\colon A \to B$ is a bi-Lipschitz map, then $\dim A = \dim B$.