Complex dynamics
Problem set 2 (due Tuesday, April 26)

1. Let (C_n) be a sequence of compact connected subsets of a metric space X. Suppose that $C_{n+1} \subset C_n$ for all $n \in \mathbb{N}$. Put
$$C = \bigcap_{n=1}^{\infty} C_n.$$
Show that C is connected.

2. Let $\Delta = \{z \in \mathbb{C} \colon |z| > 1\}$ and let $g \colon \Delta \to \mathbb{C}$ be holomorphic and injective with Laurent series
$$g(z) = z + b_0 + \sum_{n=1}^{\infty} \frac{b_n}{z^n} \quad \text{for } |z| > 1.$$
Show that
$$\text{area}(\mathbb{C}\backslash g(\Delta)) = \pi \left(1 - \sum_{n=1}^{\infty} n|b_n|^2\right)$$
so that, in particular, $|b_1| \leq 1$.

3. Let $f \colon \mathbb{D} \to \mathbb{C}$ be holomorphic and injective with $f(0) = 0$ and $f'(0) = 1$. Show that there exists a holomorphic and injective function $h \colon \mathbb{D} \to \mathbb{C}$ with $h(0) = 0$ and $h'(0) = 1$ such that
$$h(z)^2 = f(z^2).$$
Moreover, show that if f has the Taylor series expansion
$$f(z) = z + \sum_{n=2}^{\infty} a_n z^n \quad \text{for } |z| < 1,$$
then the function $g \colon \Delta \to \mathbb{C}$,
$$g(z) = \frac{1}{h\left(\frac{1}{z}\right)}$$
has the Laurent series expansion
$$g(z) = z - \frac{a_2}{2z} + O\left(\frac{1}{z^2}\right) \quad \text{as } z \to \infty.$$
Conclude, using Problem 2, that $|a_2| \leq 2$.

4. Prove the Koebe one-quarter theorem: If $f \colon \mathbb{D} \to \mathbb{C}$ is holomorphic and injective with $f(0) = 0$ and $f'(0) = 1$, then $f(\mathbb{D}) \supset D(0, \frac{1}{4})$.

   Hint. Apply the inequality $|a_2| \leq 2$ from Problem 3 to the function
$$z \mapsto \frac{wf(z)}{w - f(z)},$$
with $w \notin f(\mathbb{D})$.