Complex dynamics
Problem set 11 (due Tuesday, July 12)

1. Let $H = \{z\colon \operatorname{Im} z > 0\}$ and $S = \{z\colon |\operatorname{Im} z| < 1\}$. Compute $\lambda_H(i-1, i+1)$ and $\lambda_S(0,1)$.

   What are the hyperbolic geodesics (that is, the shortest curves with respect to the hyperbolic metric) that connects $i-1$ and $i+1$ in H and 0 and 1 in S?

2. Compute the density of the hyperbolic metric for the domains H and S of Problem 1, as well as for the domains $\mathbb{D}^* = \mathbb{D}\backslash\{0\}$, $\Delta = \{z\colon |z| > 1\}$ and $A = \{z\colon 1 < |z| < R\}$, with $1 < R < \infty$.

   Note. Your computations for A should give

$$\rho_A(z) = \frac{\pi}{\sin\left(\dfrac{\pi \log|z|}{\log R}\right)|z|\log R}$$

3. Let A be as in Problem 2 and let γ be a simple closed smooth curve with $n(\gamma, 0) = 1$. Show that

$$\ell_A(\gamma) \geq \frac{2\pi^2}{\log R}.$$

   For which curve γ do we have equality?

   Hint. Parametrize γ in the form $\gamma\colon [0,1] \to A$, $\gamma(t) = r(t)e^{i\varphi(t)}$ where $r, \varphi\colon [0,1] \to \mathbb{R}$ with $r(0) = r(1)$ and $\varphi(1) = \varphi(0) + 2\pi$.