Prof. Dr. W. Bergweiler Summer semester 2016

Complex dynamics
Problem set 10 (due Tuesday, July 5)

1. Let f be a transcendental entire function and let U be a multiply connected component of $F(f)$. Let γ be a positively oriented simple closed curve in U which is not null-homologous in U; that is, $n(\gamma, z) = 1$ for $z \in \operatorname{int}(\gamma)$ and $\operatorname{int}(\gamma) \cap (\mathbb{C} \backslash U) \neq \emptyset$.

   Show that $n(f^k \circ \gamma, 0) \to \infty$ as $k \to \infty$.

2. Let (f_k) be a sequence of functions holomorphic in a domain G. Suppose that the infinite product $\prod_{k=1}^{\infty} f_k(z)$ converges so that

   $$f(z) = \prod_{k=1}^{\infty} f_k(z)$$

   defines a function f holomorphic in G. Show that

   $$\frac{f'(z)}{f(z)} = \sum_{k=1}^{\infty} \frac{f_k'(z)}{f_k(z)},$$

   with the series on the right hand side locally uniformly convergent in $G \backslash S$, where S is the set of zeros of f.

3. Show that

   $$\frac{\pi^2}{\sin^2 \pi z} = \sum_{k=-\infty}^{\infty} \frac{1}{(z-k)^2}$$

   Hint. Show that the difference of left and right hand side is an entire function and apply Liouville's theorem to this function.

4. Show that

   $$\sin \pi z = \pi z \prod_{k=1}^{\infty} \left(1 - \frac{z^2}{k^2}\right).$$

   Hint. Use the previous problems, as well as the fact that the functions $f(z) = \sin \pi z$ and $g(z) = \pi^2 / \sin^2 \pi z$ are related by $(f'/f)' = g$.