**Complex dynamics**
**Problem set 1 (due Tuesday, April 19)**

1. Show that the Koebe function $k\colon \mathbb{D} \to \mathbb{C}\backslash(-\infty, -\frac{1}{4}]$,

$$k(z) = \frac{z}{(1-z)^2},$$

is biholomorphic.

2. Let $\Omega \subset \mathbb{R}^2$ be a domain and let $\gamma$ be a smooth simple closed curve in $\Omega$ whose interior $\operatorname{int}(\gamma)$ is also contained in $\Omega$. Let $f = (u, v)\colon \Omega \to \mathbb{R}^2$ be continuously differentiable. Green's theorem says that

$$\int_\gamma u\,dx + v\,dy = \int_{\operatorname{int}(\gamma)} \left(\frac{\partial v}{\partial x} - \frac{\partial u}{\partial y}\right) dx\,dy.$$

If you have not seen Green's theorem before, try to understand this result. (And if you have seen it before, recall what you know about it.) Show that with $f$ in complex form, that is, $f = u + iv$, and with the Wirtinger derivative

$$\frac{\partial f}{\partial \bar{z}} = \frac{1}{2}\left(\frac{\partial f}{\partial x} + i\frac{\partial f}{\partial y}\right)$$

the above formula takes the form

$$\int_\gamma f(z)dz = 2i \int_{\operatorname{int}(\gamma)} \frac{\partial f(z)}{\partial \bar{z}} dx\,dy.$$

3. Let $\gamma$ be a smooth simple closed curve. Show that

$$\operatorname{area}(\operatorname{int}(\gamma)) = \frac{1}{2i}\int_\gamma \bar{z}\,dz.$$

Here $\operatorname{area}(\cdot)$ denotes the area of a subset of $\mathbb{C}$. Let $f$ be holomorphic in a domain containing the trace of $\gamma$ and suppose that $f \circ \gamma$ is also a simple closed curve. Show that

$$\operatorname{area}(\operatorname{int}(f\circ\gamma)) = \frac{1}{2i}\int_\gamma \overline{f(z)}f'(z)\,dz.$$