Complex dynamics
Problem set 9 (due Monday, January 18)

1. Let f be entire or rational and let $U \subset \mathbb{C}$ be open with $U \cap J(f) \neq \emptyset$. Show that $(f^n|_U)$ does not have a convergent subsequence.

2. Let f be entire and let U be a multiply-connected component of $F(f)$. Show that $f^n|_U \to \infty$ as $n \to \infty$.

3. Let f be entire or rational and let $z_0 \in \mathbb{C}$ be a repelling fixed point of f with multiplier λ. Show that the normalized solution S of Schröder's functional equation (that is, the meromorphic function S satisfying $S(0) = z_0$, $S'(0) = 1$ and $S(\lambda z) = f(S(z))$ for $z \in \mathbb{C}$) is given by
$$S(z) = \lim_{n\to\infty} f^n\Big(z_0 + \frac{z}{\lambda^n}\Big).$$

4. Let
$$\sum_{k=0}^{\infty} a_k z^k$$
be a power series whose radius of convergence is 1. Suppose that $a_k \geq 0$ for all k. Show that the function $f\colon \mathbb{D} \to \mathbb{C}$,
$$f(z) = \sum_{k=0}^{\infty} a_k z^k,$$
does not have an analytic continuation to a domain G satisfying $\mathbb{D} \cup \{1\} \subset G$.

Pictures of Julia sets of the functions $f_1(z) = z^2 - \frac{1}{2}$, $f_2(z) = z^2 + i$ and $f_3(z) = z^2 + \frac{1}{4} + \frac{1}{2}i$ from problem sheet 8. In the first row, they are drawn using backward iteration, using 10^8 iterations. In the second row, the set where the iterates do not tend to infinity is drawn in red.