Prof. Dr. W. Bergweiler Winter semester 2015/16

Complex dynamics
Problem set 8 (due Monday, January 11)

For a non-constant polynomial P let the rational function $N_P\colon \widehat{\mathbb{C}} \to \widehat{\mathbb{C}}$ be defined by

$$N_P(z) = z - \frac{P(z)}{P'(z)}.$$

Newton's method consists of the iteration of N_P.

1. Let P be a polynomial of degree $d \geq 2$. Show the following:

   (i) A zero of f of multiplicity m of P is a fixed point of N_P with multiplier $1 - \dfrac{1}{m}$,

   (ii) ∞ is a fixed point of N_P with multiplier $\dfrac{d}{d-1}$,

   (iii) N_P has not fixed points other than the zeros of P and ∞.

2. Let f be a rational function with the following properties:

   (i) For all fixed points in $\mathbb{C}$ the multiplier is of the form $1 - \dfrac{1}{m}$ with some $m \in \mathbb{N}$; that is, if $z \in \mathbb{C}$ with $f(z) = z$, then there exists $m(z) \in \mathbb{N}$ such that $f'(z) = 1 - \dfrac{1}{m(z)}$,

   (ii) ∞ is a repelling fixed point of f.

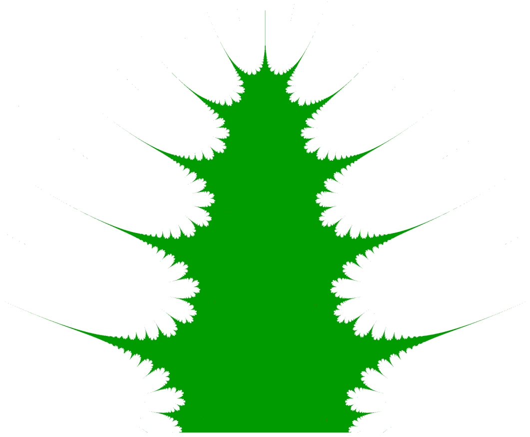
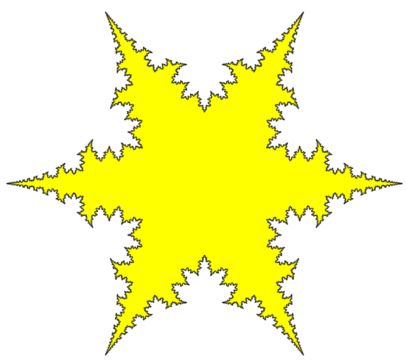
   Show that there exists a polynomial P such that $f = N_P$.

   What can be said about the form of f if condition (ii) is omitted?

3. Show that if the Julia set of a rational function is disconnected, then it has uncountably many components.

4. Draw computer pictures of the Julia sets of $f_1(z) = z^2 - \frac{1}{2}$, $f_2(z) = z^2 + i$ and $f_3(z) = z^2 + \frac{1}{4} + \frac{1}{2}i$ using backward iteration; that is, by finding a point in the Julia set and using that the Julia set is the closure of its backward orbit.

   Compare these pictures with pictures of the attracting basin of ∞; that is, the sets where the iterates of f_j tend to ∞.

Fatou and Julia sets of $f(z) = -i\left(\cos\sqrt{iz} + 1 + \pi^2\right)$ and $g(z) = z\dfrac{26z^6 - 4}{25z^6 + 5}$

Merry Christmas and Happy New Year!