Complex dynamics
Problem set 7 (due Monday, December 14)

1. For $c \in \mathbb{C}$ let $f_c \colon \mathbb{C} \to \mathbb{C}$, $f_c(z) = z^2 - c$. Show that

   (i) $J(f_c) \subset \mathbb{R}$ for $c \geq 2$,

   (ii) $J(f_2) = [-2, 2]$.

2. Let $d \geq 2$, let $a_1, \dots, a_d$ be positive real numbers and let $b_1, \dots, b_d$ be distinct real numbers. Define the rational function f by

$$f(z) = \sum_{k=1}^{d} \frac{a_k}{z - b_k}.$$

   Show that $J(f) \subset \mathbb{R} \cup \{\infty\}$.

3. Let f be a rational or entire function. Show that there exists a point in the Julia set of f whose forward orbit is dense in the Julia set of f; that is, there exists $z \in J(f)$ such that $\overline{O^+(z)} = J(f)$.

   Hint. Show first that if $(U_k)_{k \geq 0}$ is a sequence of domains intersecting $J(f)$, then there exists an increasing sequence $(n_k)_{k \geq 1}$ in $\mathbb{N}$ and a sequence $(V_k)_{k \geq 1}$ of domains satisfying $\overline{V_{k+1}} \subset V_k \subset U_0$ such that $f^{n_k}(V_k) \subset U_k$ for all $k \geq 1$. Show then that for a suitably chosen sequence (U_k) a point z_0 in $\bigcap_{k=1}^{\infty} \overline{V_k}$ has the required property.

4. Show that a perfect subset of a complete metric space is uncountable.