Prof. Dr. W. Bergweiler Winter semester 2015/16

Complex dynamics
Problem set 6 (due Monday, December 7)

1. Let G and H be the domains given below. Find a biholomorphic function $f\colon G \to H$.
   (a) $G = \mathbb{D}$, $H = \{z \in \mathbb{C}\colon \operatorname{Im} z > 0\}$,
   (b) $G = \{z \in \mathbb{C}\colon \operatorname{Im} z > 0, \operatorname{Re} z > 0\}$, $H = \{z \in \mathbb{C}\colon \operatorname{Im} z > 0\}$,
   (c) $G = \{z \in \mathbb{C}\colon \operatorname{Im} z > 0, \operatorname{Re} z > 0\}$, $H = \mathbb{D}$,
   (d) $G = \{z \in \mathbb{C}\colon \operatorname{Im} z > 0, |z| < 1\}$, $H = \mathbb{D}$,
   (e) $G = \{z \in \mathbb{C}\colon |\operatorname{Im} z| < 1\}$, $H = \mathbb{D}$,

2. Let $G, H \subset \widehat{\mathbb{C}}$ be domains satisfying $\overline{H} \subset G$ and let $f\colon G \to \widehat{\mathbb{C}}$ be meromorphic. Suppose that $f(H) \subset H$ and that $\left\{ f^n\big|_H \colon n \in \mathbb{N} \right\}$ is normal. Suppose also that $f\colon H \to H$ is not bijective. Show that there exists $a \in \overline{H}$ satisfying $f(a) = a$ such that $f^n \to a$ locally uniformly in H.

3. Let f and g be entire (or rational) functions and let $T\colon \widehat{\mathbb{C}} \to \widehat{\mathbb{C}}$ be a homeomorphism; that is, T is bijective and T and T^{-1} are continuous. If f and g are entire, assume in addition that $T(\infty) = \infty$. Suppose that $T(f(z)) = g(T(z))$ for all $z \in \mathbb{C}$ (or $z \in \widehat{\mathbb{C}}$). Show that $F(f) = T(F(g))$ und $J(f) = T(J(g))$.

4. Let $a, b \in \mathbb{D}$ and $f, g\colon \mathbb{C} \to \mathbb{C}$, $f(z) = az$, $g(z) = bz$. Show that f and g are conjugate; that is, there exists a homeomorphism $T : \mathbb{C} \to \mathbb{C}$ satisfying $T \circ f = g \circ T$.

   Hint. Consider maps of the form $T(re^{it}) = r^\gamma e^{i(t+\delta \log r)}$ with $\gamma, \delta \in \mathbb{R}$.