Prof. Dr. W. Bergweiler Winter semester 2015/16

Complex dynamics
Problem set 5 (due Monday, November 30)

1. Show that every injective entire function f is of the form $f(z) = az + b$ with $a, b \in \mathbb{C}$ and $a \neq 0$.

2. Let $\mathcal{F}$ be a set of functions holomorphic and injective in a domain G and let $c \in \mathbb{C}$ and $z_0 \in G$. Show that $\mathcal{F}$ is normal if one of the following conditions holds:

   (a) $f(z) \neq c$ for all $f \in \mathcal{F}$ and all $z \in G$;

   (b) $f(z_0) = 0$ and $|f'(z_0)| \leq 1$ for all $f \in \mathcal{F}$.

3. Let $\mathcal{F}$ be a set of functions holomorphic in a domain G and let $N \in \mathbb{N}$. Suppose that $f(z) \neq 1$ for all $f \in \mathcal{F}$ and all $z \in G$. Suppose that if $f \in \mathcal{F}$, then f has at most N zeros, counting multiplicities. Show that $\mathcal{F}$ is normal.

4. Let f be a non-constant entire function which is not a translation (i.e., not of the form $f(z) = z + c$ with $c \in \mathbb{C}$). Show that $f \circ f$ has a fixed point.

   Hint: Apply Picard's theorem to the function

$$h(z) = \frac{f(f(z)) - z}{f(z) - z}.$$