Complex dynamics
Problem set 4 (due Monday, November 23)

1. Let $f\colon \widehat{\mathbb{C}} \to \widehat{\mathbb{C}}$ be the rational function defined by

   (a) $f_n(z) = nz^n$

   (b) $f_n(z) = \dfrac{z^3}{1+nz^2}$

   For which $z \in \widehat{\mathbb{C}}$ is $\{f_n\colon n \in \mathbb{N}\}$ normal in z.

2. Let $\mathcal{F}$ be a set of functions holomorphic in a domain G. Let $\mathcal{D} = \{f'\colon f \in \mathcal{F}\}$ be the family of derivatives and $\mathcal{A} = \{F\colon G \to \mathbb{C}, F \text{ is holomorphic, } F' \in \mathcal{F}\}$ be the family of anti-derivatives.

   Which of the following implications are correct:

   (a) If $\mathcal{F}$ is normal, then $\mathcal{D}$ is normal (?)

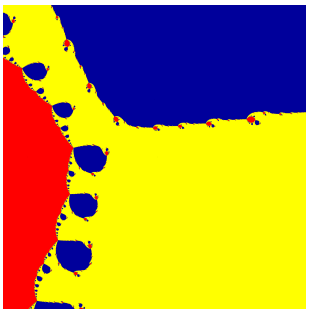
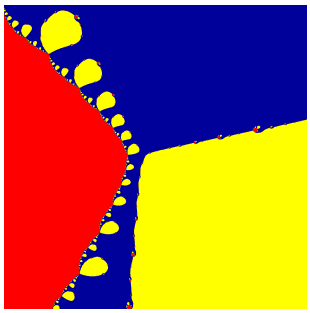
   (b) If $\mathcal{F}$ is normal, then $\mathcal{A}$ is normal (?)

   (c) If $\mathcal{D}$ is normal, then $\mathcal{F}$ is normal (?)

   (d) If $\mathcal{A}$ is normal, then $\mathcal{F}$ is normal (?)

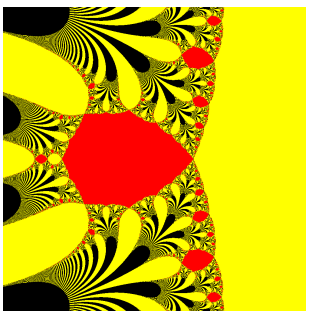
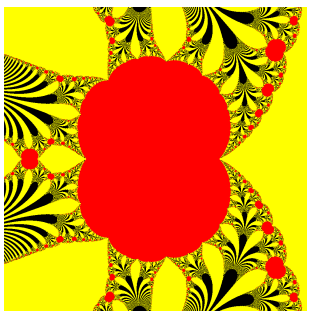
For the next two problems, let d, m and y be the day, month and year of your birth.

3. Let $z_1 = m$, $z_2 = id$ and $z_3 = y - 2000$. Put $p(z) = \prod_{j=1}^{3}(z - z_j)$ and let $f(z) = z - p(z)/p'(z)$ be the function from Newton's method. Produce computer pictures of the sets $\{z\colon \lim_{n\to\infty} f^n(z) = z_j\}$.

Pictures with the dates of Walter Bergweiler and Daniel Sommerfeld

4. Let $f(z) = z^{m+1}\exp\big(\big(m + \frac{d}{16}\big)(1-z)\big)$. Show that 0 and 1 are fixed points and that $f'(0) = 0$ and $|f'(1)| < 1$. Produce computer pictures of the sets $\{z\colon \lim_{n\to\infty} f^n(z) = 0\}$ and $\{z\colon \lim_{n\to\infty} f^n(z) = 1\}$.

Pictures with the dates of Walter Bergweiler and Daniel Sommerfeld. The range shown is $-1 \le \operatorname{Re} z \le 5$, $|\operatorname{Im} z| \le 3$.