Prof. Dr. W. Bergweiler Winter semester 2015/16

Complex dynamics
Problem set 11 (due Monday, February 1)

1. Let $a > 0$ and
$$f(z) = \frac{1}{a+1}\, z^2(z+a).$$
Determine the behavior of the critical points of f under iteration. Distinguish the cases $0 < a < 3$, $a = 3$ and $a > 3$. Conclude that f has no attracting periodic points besides the superattracting fixed points 0 and ∞.

2. Let f be entire or rational with a superattracting fixed point $z_0 \in \mathbb{C}$. Let $\phi\colon V \to \mathbb{C}$ be the solution of Böttcher's functional equation $\phi(f(z)) = \phi(z)^m$ as in Theorem 6.7, with $m \in \mathbb{N}$ and a_m, b_1 as there. Suppose that $f(z) \neq z_0$ for all $z \in A^*(z_0)$ and $\infty \notin A^*(z_0)$.

   Show that ϕ has a holomorphic continuation to $A^*(z_0)$.

   As in the proof of Theorem 6.7 you may assume that $z_0 = 0$, $a_m = 1$, $b_1 = 1$.

3. Let f, g, ϕ be holomorphic in a neighborhood of 0 with Taylor series expansions
$$\begin{aligned} f(z) &= z + a_2 z^2 + a_3 z^3 + \dots\,, \\ g(z) &= z + b_2 z^2 + b_3 z^3 + \dots\,, \\ \phi(z) &= c_1 z + c_2 z^2 + c_3 z^3 + \dots\,, \end{aligned}$$
with $a_2, b_2, c_1 \neq 0$. Suppose that $f(\phi(z)) = \phi(g(z))$ for z near 0. Show that $b_2 = c_1 a_2$ and $b_3/b_2^2 = a_3/a_2^2$.

4. Let f be as in Problem 3. Let $F(z) = 1/(z - f(z))$. Show that $\operatorname{res}(F, 0) = a_3/a_2^2$. (Here $\operatorname{res}(F, 0)$ is the residue of F at the pole 0.)

Pictures for Problem 4 of the previous problem set 10: The colored regions show the set of a-values where Newton's method for $P(z) = (z-1)(z+1)(z-a)$ with starting value $a/3$ converges to one of the zeros of P. In the black set, Newton's method does not converge to a zero of P. The range shown is $|\operatorname{Im} z| \le 6$ and $|\operatorname{Re} z| \le 6$ in the left picture and $|\operatorname{Im} z - 4.5| \le 0.2$ and $|\operatorname{Re} z| \le 0.2$ in the right picture.