Complex dynamics
Problem set 1 (due Tuesday, November 3)

1. Let $a, b \in \mathbb{C}$, $a \neq 0$, and $f\colon \mathbb{C} \to \mathbb{C}$, $f(z) = az + b$. Find a "closed-form expression" for the iterates $f^n(z)$. Discuss the limiting behavior as $n \to \infty$, distinguishing the cases $|a| < 1$ and $|a| > 1$.

2. Let $G \subset \mathbb{C}$ be a domain and let $f\colon G \to \mathbb{C}$ be holomorphic. Suppose that there exists $\xi \in G$ such that $f(\xi) = \xi$ and $|f'(\xi)| < 1$. Show that there exists $\varepsilon > 0$ such that $\lim_{n\to\infty} f^n(z) = \xi$ for all $z \in G$ satisfying $|z - \xi| < \varepsilon$.

3. Let p be a polynomial of degree at least 2 and let $z_1, z_2, \ldots, z_m$ be the fixed points of p. (A fixed point is a solutions of the equation $p(z) = z$.) Suppose that $p'(z_j) \neq 1$ for all $j \in \{1, \ldots, m\}$. Show that
$$\sum_{j=1}^{m} \frac{1}{1 - p'(z_j)} = 0.$$

   Hint: Use the residue theorem.