Analytic number theory
Problem set 8 (due Friday, June 22)

1. Complete the proof of Theorem 8.2 by showing that if $\operatorname{Re} z > 0$, then

$$\int_0^n \left(1 - \frac{t}{n}\right)^n t^{z-1} dt \to \Gamma(z)$$

as $n \to \infty$.

2. Show that

$$\Gamma(z) = \frac{e^{-\gamma z}}{z \prod_{k=1}^{\infty} \left(1 + \frac{z}{k}\right) e^{-z/k}} = \frac{e^{-\gamma z}}{z} \prod_{k=1}^{\infty} \frac{k e^{z/k}}{k+z},$$

with γ as in Problem 1 of Problem set 5.

3. Show that

$$\sin(\pi z) = \pi z \prod_{k \in \mathbb{Z} \setminus \{0\}} \left(1 + \frac{z}{k}\right) e^{-z/k} = \pi z \prod_{k=1}^{\infty} \left(1 - \frac{z^2}{k^2}\right).$$

Hint. Consider the logarithmic derivative of both sides.

4. Show that

$$\Gamma(z)\Gamma(1-z) = \frac{\pi}{\sin(\pi z)}.$$

5. Show that

$$\Gamma(z)\Gamma\left(z + \frac{1}{2}\right) = 2^{1-2z}\sqrt{\pi}\,\Gamma(2z).$$

Hint. Use Theorem 8.2 and Stirling's formula.