Analytic number theory
Problem set 7 (due Friday, June 15)

1. The numbers E_n defined by
$$\frac{1}{\cosh z} = \sum_{n=0}^{\infty} \frac{E_n}{n!} z^n$$
are called *Euler numbers*. Here
$$\cosh z = \frac{1}{2}(e^z + e^{-z})$$
is the hyperbolic cosine.

   We have $\cosh z = \cos iz$ and $\cos z = \cosh iz$. The power series for $1/\cosh z$ converges in some neighborhood of 0, in fact for $|z| < 2\pi$.

   Show that $E_n = 0$ if n is odd and that
$$E_{2m} = -\sum_{k=0}^{m-1} \binom{2m}{2k} E_{2k}$$
for $m \in \mathbb{N}$. Compute E_0, E_1 and E_2.

2. Show that
$$\frac{\pi}{\cos(\pi z)} = \pi + \sum_{n\in\mathbb{Z}} (-1)^{n+1} \left(\frac{1}{z - \left(n + \frac{1}{2}\right)} + \frac{1}{n + \frac{1}{2}} \right).$$

3. Let $m \in \mathbb{N}$. Show that
$$\sum_{n=0}^{\infty} \frac{(-1)^n}{(2n+1)^{2m+1}} = \frac{(-1)^m E_{2m} \pi^{2m+1}}{4^{m+1}(2m)!}.$$

4. Let $\mu\colon \mathbb{N} \to \{-1, 0, 1\}$ be as in Problem 3 of Problem set 4 and define $M\colon \mathbb{R} \to \mathbb{R}$,
$$M(x) = \sum_{n\leq x} \mu(n).$$
Let $\alpha \in [\frac{1}{2}, 1)$ and suppose that for every $\varepsilon > 0$ we have $M(x) = \mathcal{O}(x^{\alpha+\varepsilon})$ as $x \to \infty$. Show that $\zeta(z) \neq 0$ for $z \in \mathbb{H}_\alpha$.

   Hint. Consider the proof of Lemma 5.4. Use also Problem 3 of Problem set 4.