Analytic number theory
Problem set 6 (due Friday, June 8)

1. Show that
$$\pi(x) - \mathrm{Li}(x) = \frac{\theta(x) - x}{\log x} + \int_2^x \frac{\theta(t) - t}{t(\log t)^2} dt + \frac{2}{\log 2}$$
and
$$\theta(x) - x = (\pi(x) - \mathrm{Li}(x)) \log x + \int_2^x \frac{\pi(t) - \mathrm{Li}(t)}{t} dt - 2$$
for $x \geq 2$.

   Hint. Let $\mathbb{P} = \{p_1, p_2, \ldots\}$ and write the integrals involving $\theta(t)$ and $\pi(t)$ as a sum of integrals from p_k to p_{k+1}, plus an integral from p_n to x, with a suitable n.

2. Let $\frac{1}{2} \leq \alpha < 1$. Show that the following three statements are equivalent:
   (a) $\pi(x) - \mathrm{Li}(x) = \mathcal{O}(x^\alpha \log x)$ as $x \to \infty$;
   (b) $\theta(x) - x = \mathcal{O}(x^\alpha (\log x)^2)$ as $x \to \infty$;
   (c) $\psi(x) - x = \mathcal{O}(x^\alpha (\log x)^2)$ as $x \to \infty$.

3. Prove Abel's limit theorem: *If*
$$\sum_{n=0}^{\infty} a_n$$
is a convergent series of complex numbers and $f\colon [0,1] \to \mathbb{C}$ *is defined by*
$$f(x) = \sum_{n=0}^{\infty} a_n x^n,$$
then
$$\lim_{x \to 1} f(x) = \sum_{n=0}^{\infty} a_n.$$

   Hint. Consider the generating function of the sequence $(s_n)_{n \geq 0}$ given by
$$s_n = \sum_{k=0}^{n} a_k$$
as in Problem 3 of Problem set 1. The argument simplifies if you assume, without loss of generality, that
$$\sum_{n=0}^{\infty} a_n = 0.$$

4. Let (a_n) be a sequence of complex numbers and put $\varphi\colon \mathbb{R} \to \mathbb{R}$,

$$\varphi(x) = \sum_{n \le x} a_n = \sum_{n=1}^{\lfloor x \rfloor} a_n,$$

with $\varphi(x) = 0$ for $x < 1$. Let $z \in \mathbb{C}$ with $\operatorname{Re} z > 0$ and suppose that

$$\sum_{n=1}^{\infty} \frac{a_n}{n^z}$$

converges. Show that $\varphi(N) = o(N^z)$ as $N \to \infty$.

Remark. In the proof of Lemma 5.4 this was shown if the a_n are non-negative real numbers. (The condition $\operatorname{Re} z > 0$ was not stated in the lecture, but it is also required for the proof given there.) It follows from this problem that the conclusion of Lemma 5.4 also holds without the assumption that the a_n are non-negative.

Hint. Put

$$S(N) = \sum_{n=1}^{N} \frac{a_n}{n^z}$$

and, proceeding similar as in the proof of Lemma 5.4, show that

$$\varphi(N) = S(N)N^z + \sum_{n=1}^{N-1} S(n)(n^z - (n+1)^z).$$

Similarly as in the previous problem, you may assume that

$$\sum_{n=1}^{\infty} \frac{a_n}{n^z} = 0.$$