Analytic number theory
Problem set 5 (due Friday, June 1)

1. The Euler-Mascheroni constant γ is defined by

$$\gamma = \lim_{n\to\infty}\left(\sum_{k=1}^{n}\frac{1}{k} - \log(n)\right).$$

   Show that

$$\zeta(z) = \frac{1}{z-1} + \gamma + \mathcal{O}(z-1)$$

   as $z \to 1$.

   Hint. See Theorem 4.1 and its proof.

2. Show that

$$(z-1)\zeta(z) = \sum_{n=1}^{\infty}\left(\frac{z-n}{n^z} + \frac{n}{(n+1)^z}\right)$$

   for $z \in \mathbb{H}_0$.

3. Let $\mathbb{P} = \{p_k \colon k \in \mathbb{N}\}$, with $p_1 < p_2 < p_3 < p_4 < \dots$ so that $p_1 = 2$, $p_2 = 3$, $p_3 = 5$, $p_4 = 7$, etc.

   Show that $p_k \sim k \log k$ as $k \to \infty$.

4. Show that

$$\sum_{p\in\mathbb{P}}\frac{1}{p^z} = \sum_{n=1}^{\infty}\frac{\mu(n)}{n}\log\zeta(nz)$$

   for $z \in \mathbb{H}_1$, where μ is as in Problem 3 of Problem set 4.

   Hint. Use the formula for μ obtained in Problem 4 of Problem set 4 and the equation

$$\log\zeta(x) = \sum_{p\in\mathbb{P}}\sum_{k=1}^{\infty}\frac{1}{k}\cdot\frac{1}{p^{kx}}.$$

   of Theorem 4.5.