Analytic number theory
Problem set 4 (due Friday, May 25)

1. Show that the series
$$\sum_{n=1}^{\infty} \frac{(-1)^n}{n^z}$$
converges for all $\mathbb{H}_0$ and that
$$(2^{1-z}-1)\zeta(z) = \sum_{n=1}^{\infty} \frac{(-1)^n}{n^z}$$
for all $z \in \mathbb{H}_1$.

   Hint. Show that if n is large enough, then
$$\left| \frac{(-1)^{n-1}}{(n-1)^z} + \frac{(-1)^n}{n^z} \right| \leq 2|z| \frac{1}{n^{\operatorname{Re} z}} \log\left(\frac{n}{n-1}\right) \leq 4|z| \frac{1}{n^{1+\operatorname{Re} z}}.$$

2. Show that
$$\sum_{p\in\mathbb{P}} \frac{1}{p}$$
diverges.

3. Define $\mu\colon \mathbb{N} \to \{-1,0,1\}$ by $\mu(1)=1$ and, for $n \geq 2$, by $\mu(n) = (-1)^k$ if n is the product of k distinct primes and $\mu(n)=0$ otherwise. Thus $\mu(n)=0$ if n is divisible by p^n for some $p \in \mathbb{P}$ and $k \in \mathbb{N}\backslash\{1\}$. For example, $\mu(2)=-1$, $\mu(3)=-1$, $\mu(4)=0$, $\mu(5)=-1$, $\mu(6)=1$.

   Show that
$$\frac{1}{\zeta(z)} = \sum_{n=1}^{\infty} \frac{\mu(n)}{n^z}$$
for all $z \in \mathbb{H}_1$.

4. Let $a, b\colon \mathbb{N} \to \mathbb{C}$ and $r \in \mathbb{R}$. Suppose that the series
$$f(z) = \sum_{n=1}^{\infty} \frac{a(n)}{n^z} \quad \text{and} \quad g(z) = \sum_{n=1}^{\infty} \frac{b(n)}{n^z}$$
converge absolutely for $z \in \mathbb{H}_r$.

   Show that
$$f(z)g(z) = \sum_{n=1}^{\infty} \frac{c(n)}{n^z}$$
for $z \in \mathbb{H}_r$, where $c\colon \mathbb{N} \to \mathbb{C}$,
$$c(n) = \sum_{\substack{d\in\mathbb{N}\\ d|n}} a(d)\, b\!\left(\frac{n}{d}\right).$$

Use this to show that, with μ as in Problem 2,

$$\sum_{d|n} \mu(d) = \begin{cases} 1, & n = 1, \\ 0, & n \geq 2, \end{cases}$$

for $n \in \mathbb{N}$.

Show also that

$$\zeta(z)^2 = \sum_{n=1}^{\infty} \frac{\tau(n)}{n^z}$$

for $z \in \mathbb{H}_1$, where $\tau(n)$ is the number of divisors of n.