Prof. Dr. W. Bergweiler Summer semester 2018

Analytic number theory
Problem set 3 (due Friday, May 11)

1. For $n \in \mathbb{N}$ denote by c_n the number of possibilities to divide the set $\{1, 2, \cdots, n\}$ into disjoint subsets. For example, $c_n = 5$ since

$$\begin{aligned}\{1,2,3\} &= \{1,2,3\},\\ \{1,2,3\} &= \{1,2\} \cup \{3\},\\ \{1,2,3\} &= \{1,3\} \cup \{2\},\\ \{1,2,3\} &= \{2,3\} \cup \{1\} \quad \text{and}\\ \{1,2,3\} &= \{1\} \cup \{2\} \cup \{3\}.\end{aligned}$$

Note that c_n is the number of equivalence relations on a set with n elements.

Put $c_0 = 1$. Show that the c_n satisfy the recursion

$$c_{n+1} = \sum_{k=0}^{n} \binom{n}{k} c_k$$

2. Show that $f\colon \mathbb{R} \to \mathbb{R}$, $f(x) = \exp(e^x - 1)$ satisfies

$$f(x) = \sum_{n=0}^{\infty} \frac{c_n}{n!} x^n$$

for $x \in \mathbb{R}$, with c_n as in Problem 1.

Hint. Show that $f'(x) = e^x f(x)$.

3. Let $V\colon [0,\infty) \to [0,\infty)$, $V(x) = xe^x$. Show that V is bijective and that the inverse function W of V satisfies the following:

   (a) $W(x) \sim \log x$ as $x \to \infty$;

   (b) $W(x) = \log x - \log\log x + o(1)$ as $x \to \infty$;

   (c) $W(x) \le \log x$ for $x \ge e$;

   (d) $W(x) \ge \log x - \log\log x$ for $x \ge e$.

4. Let c_n be as in Problem 1 and let W be the function from Problem 3. Show that

$$c_n \le \frac{n!}{W(n)^n} \exp\left(\frac{n}{W(n)} - 1\right)$$

for all $n \in \mathbb{N}$.

5. Let c_n be as in Problem 1 and let W be the function from Problem 3. Let

$$d_n := \frac{n!}{W(n)^n} \exp\left(\frac{n}{W(n)} - 1\right)$$

be the upper bound for c_n given in Problem 4.

Show that

$$c_n \sim \frac{1}{\sqrt{2\pi n W(n)}} d_n \sim \frac{n^n}{W(n)^{n+1/2}} \exp\left(\frac{n}{W(n)} - n - 1\right).$$

Hint. Use the saddle point method with $g(z) = e^z - 1$ as described in the lecture. Show that, with $a(r)$ and $b(r)$ as defined in the lecture, we have

$$g(re^{it}) = g(r) + a(r)it - \frac{1}{2}b(r)t^2 + R(t)t^3,$$

where

$$|R(t)| \leq Cr^3 e^r \leq Crb(r) \quad \text{for } |t| \leq \pi$$

with some constant C.