Prof. Dr. W. Bergweiler Summer semester 2018

Analytic number theory
Problem set 2 (due Friday, April 27)

1. Let $(a_n)_{n\geq 0}$ be defined by $a_0 = 1$ and $a_{n+1} = 2a_n - n - 1$ for $n \geq 0$.

   Determine the generating function of (a_n) and use this to find a closed-form expression for a_n.

2. Let $(a_n)_{n\geq 0}$ be defined by $a_0 = 1$ and

   $$a_{n+1} = \sum_{k=0}^{n} a_k a_{n-k}$$

   for $n \geq 0$.

   Determine the generating function of (a_n) and use this to find a closed-form expression for a_n.

   Remark. For $\alpha \in \mathbb{R}$ and $|x| < 1$ we have the expansion

   $$(1+x)^\alpha = \sum_{k=0}^{\infty} \binom{\alpha}{k} x^k$$

   where

   $$\binom{\alpha}{k} = \frac{\alpha \cdot (\alpha-1) \cdot (\alpha-2) \cdot \ldots \cdot (\alpha-k+1)}{k!} = \frac{\alpha}{1} \cdot \frac{\alpha-1}{2} \cdot \frac{\alpha-2}{3} \cdot \ldots \cdot \frac{\alpha-k+1}{k}.$$

3. Show that the infinite product

   $$\prod_{k=1}^{\infty} \cos\left(\frac{z}{2^k}\right)$$

   converges for all $z \in \mathbb{C}$.

   Use the formula $\sin 2z = 2\sin z \cos z$ to find a closed-form expression for the partial products

   $$\prod_{k=1}^{n} \cos\left(\frac{z}{2^k}\right)$$

   with $n \in \mathbb{N}$. Use this to determine the value of the infinite product.