

```

> f:=x*y/(1+x^4+y^2+x^2);

```

$$f := \frac{xy}{1+x^4+y^2+x^2}$$

```

> with(linalg):
Warning, the protected names norm and trace have been redefined and unprotected
> gr:=convert(grad(f,[x,y]),set);

```

$$gr := \left\{ \frac{x}{1+x^4+y^2+x^2} - \frac{2xy^2}{(1+x^4+y^2+x^2)^2}, \frac{y}{1+x^4+y^2+x^2} - \frac{xy(4x^3+2x)}{(1+x^4+y^2+x^2)^2} \right\}$$

```

> s:=solve(gr,{x,y});
s := {x=0,y=0}, {y=RootOf(_Z^2-3),x=1}, {y=RootOf(_Z^2-3),x=-1},
      {y=1,x=RootOf(_Z^2+1)}, {x=RootOf(_Z^2+1),y=-1}
> s1:=s[1];

```

$$s1 := \{x=0, y=0\}$$

```

> s2:=allvalues(s[2]);

```

$$s2 := \{x=1, y=\sqrt{3}\}, \{x=1, y=-\sqrt{3}\}$$

```

> s3:=allvalues(s[3]);

```

$$s3 := \{x=-1, y=\sqrt{3}\}, \{x=-1, y=-\sqrt{3}\}$$

```

> he:=hessian(f,[x,y]);
he :=

```

$$\begin{bmatrix} -2\frac{y(4x^3+2x)}{(1+x^4+y^2+x^2)^2} + \frac{2xy(4x^3+2x)^2}{(1+x^4+y^2+x^2)^3} - \frac{xy(12x^2+2)}{(1+x^4+y^2+x^2)^2}, \\ \frac{1}{1+x^4+y^2+x^2} - \frac{x(4x^3+2x)}{(1+x^4+y^2+x^2)^2} - \frac{2y^2}{(1+x^4+y^2+x^2)^2} + \frac{4xy^2(4x^3+2x)}{(1+x^4+y^2+x^2)^3} \\ \frac{1}{1+x^4+y^2+x^2} - \frac{x(4x^3+2x)}{(1+x^4+y^2+x^2)^2} - \frac{2y^2}{(1+x^4+y^2+x^2)^2} + \frac{4xy^2(4x^3+2x)}{(1+x^4+y^2+x^2)^3}, \\ -6\frac{xy}{(1+x^4+y^2+x^2)^2} + \frac{8xy^3}{(1+x^4+y^2+x^2)^3} \end{bmatrix}$$

```

> subs(s1,matrix(he));

```

$$\begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}$$

```

> eigenvals(%);

```

$$1, -1$$

```

> eigenvals(subs(s2[1],matrix(he)));

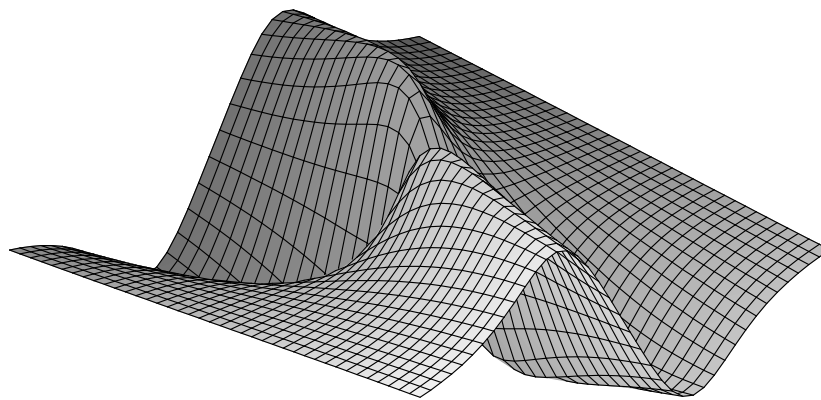
```

$$-\frac{2}{9}\sqrt{3} + \frac{1}{3}, -\frac{2}{9}\sqrt{3} - \frac{1}{3}$$

```

> evalf(%);
                                -.0515668462, -.7182335128
> eigenvals(subs(s2[2],matrix(he)));
                                 $\frac{2}{9}\sqrt{3} + \frac{1}{3}, \frac{2}{9}\sqrt{3} - \frac{1}{3}$ 
> evalf(%);
                                .7182335128, .0515668462
> eigenvals(subs(s3[1],matrix(he)));
                                 $\frac{2}{9}\sqrt{3} + \frac{1}{3}, \frac{2}{9}\sqrt{3} - \frac{1}{3}$ 
> eigenvals(subs(s3[2],matrix(he)));
                                 $-\frac{2}{9}\sqrt{3} + \frac{1}{3}, -\frac{2}{9}\sqrt{3} - \frac{1}{3}$ 
> plot3d(f,x=-5..5,y=-5..5,grid=[40,40]);

```



```

> minimize(f,x=-4..4,y=-4..4);
                                minimize( $\frac{xy}{1+x^4+y^2+x^2}, x=-4..4, y=-4..4$ )
> maximize(f,x=-4..4,y=-4..4);
                                maximize( $\frac{xy}{1+x^4+y^2+x^2}, x=-4..4, y=-4..4$ )
>
> f:=(1-x^2+3*y^4)/sqrt(x^2+2*y^2+1);

```

```

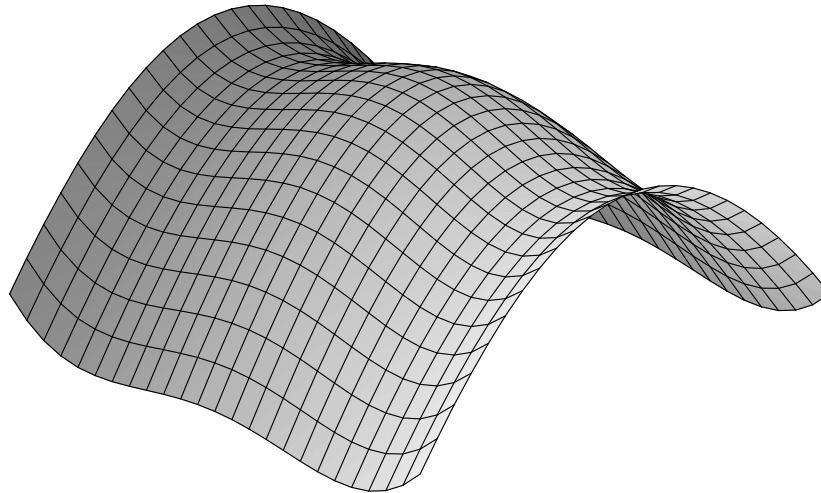
|
|                                      $f := \frac{1 - x^2 + 3y^4}{\sqrt{x^2 + 2y^2 + 1}}$ 
|
| > gr := convert(grad(f, [x, y]), set);
|
|                                      $gr := \left\{ -2 \frac{x}{\sqrt{x^2 + 2y^2 + 1}} - \frac{(1 - x^2 + 3y^4)x}{(x^2 + 2y^2 + 1)^{(3/2)}, 12 \frac{y^3}{\sqrt{x^2 + 2y^2 + 1}} - \frac{2(1 - x^2 + 3y^4)y}{(x^2 + 2y^2 + 1)^{(3/2)}} \right\}$ 
|
| > s := solve(gr, {x, y});
|
| s := {x=0, y=0},
|
|      {x=0, y=RootOf(-RootOf(_Z^2 + 2_Z - 1, label = _L72) + 3_Z^2, label = _L70)},
|
|      {y=0, x=RootOf(_Z^2 + 3, label = _L71)}, {x=RootOf(_Z^2 + 2), y=RootOf(1 + 3_Z^2)}
|
| > s1 := s[1];
|
|                                     s1 := {x=0, y=0}
|
| > s2 := allvalues(s[2]);
|
| s2 := {x=0, y= $\frac{1}{3}\sqrt{-3 + 3\sqrt{2}}$ }, {x=0, y= $-\frac{1}{3}\sqrt{-3 + 3\sqrt{2}}$ }, {x=0, y= $\frac{1}{3}\sqrt{-3 - 3\sqrt{2}}$ },
|
|      {x=0, y= $-\frac{1}{3}\sqrt{-3 - 3\sqrt{2}}$ }
|
| > evalf(%);
|
| {y=.3715793150, x=0.}, {y=-.3715793150, x=0.}, {y=.8970718219 I, x=0.},
|
|      {x=0., y=-.8970718219 I}
|
| > he := hessian(f, [x, y]);
|
| he :=
|
|       $\left[ -2 \frac{1}{\sqrt{x^2 + 2y^2 + 1}} + \frac{4x^2}{(x^2 + 2y^2 + 1)^{(3/2)}} + \frac{3(1 - x^2 + 3y^4)x^2}{(x^2 + 2y^2 + 1)^{(5/2)}} - \frac{1 - x^2 + 3y^4}{(x^2 + 2y^2 + 1)^{(3/2)}, \right.$ 
|
|       $\left. -12 \frac{y^3 x}{(x^2 + 2y^2 + 1)^{(3/2)}} + \frac{4xy}{(x^2 + 2y^2 + 1)^{(3/2)}} + \frac{6(1 - x^2 + 3y^4)yx}{(x^2 + 2y^2 + 1)^{(5/2)}} \right]$ 
|
|       $\left[ -12 \frac{y^3 x}{(x^2 + 2y^2 + 1)^{(3/2)}} + \frac{4xy}{(x^2 + 2y^2 + 1)^{(3/2)}} + \frac{6(1 - x^2 + 3y^4)yx}{(x^2 + 2y^2 + 1)^{(5/2)}, \right.$ 
|
|       $\left. 36 \frac{y^2}{\sqrt{x^2 + 2y^2 + 1}} - \frac{48y^4}{(x^2 + 2y^2 + 1)^{(3/2)}} + \frac{12(1 - x^2 + 3y^4)y^2}{(x^2 + 2y^2 + 1)^{(5/2)}} - \frac{2(1 - x^2 + 3y^4)}{(x^2 + 2y^2 + 1)^{(3/2)}} \right]$ 
|
| > subs(s[1], matrix(he));
|
|                                      $\begin{bmatrix} -3 & 0 \\ 0 & -2 \end{bmatrix}$ 
|
| > subs(s2[1], matrix(he));
|

```

```

[
  [
    -2 * 1 / sqrt(1/3 + 2/3 * sqrt(2)) - (1 + 1/27 * (-3 + 3 * sqrt(2))^2) / ((1/3 + 2/3 * sqrt(2))^(3/2)), 0
  ],
  [
    0, 4 * (-3 + 3 * sqrt(2)) / sqrt(1/3 + 2/3 * sqrt(2)) - 16 * (-3 + 3 * sqrt(2))^2 / ((1/3 + 2/3 * sqrt(2))^(3/2)) + 4/3 * (1 + 1/27 * (-3 + 3 * sqrt(2))^2) * (-3 + 3 * sqrt(2)) / ((1/3 + 2/3 * sqrt(2))^(5/2)) - 2 * (1 + 1/27 * (-3 + 3 * sqrt(2))^2) / ((1/3 + 2/3 * sqrt(2))^(3/2))
  ]
]
> simplify(%);
[
  [
    -6 * (4 + sqrt(2)) / ((1 + 2 * sqrt(2)) * sqrt(3 + 6 * sqrt(2))), 0
    ],
  [
    0, 72 * (-2 + 3 * sqrt(2)) / ((1 + 2 * sqrt(2))^2 * sqrt(3 + 6 * sqrt(2)))
    ]
]
> eigenvals(%);
624/49 * sqrt(3 + 6 * sqrt(2)) - 408/49 * sqrt(3 + 6 * sqrt(2)) * sqrt(2), -8/7 * sqrt(3 + 6 * sqrt(2)) + 2/7 * sqrt(3 + 6 * sqrt(2)) * sqrt(2)
> evalf(%);
3.25072781, -2.503775748
> plot3d(f, x=-0.5..0.5, y=-0.5..0.5);

```



```
> maximize(f,x=-0.5..0.5,y=-0.5..0.5);
```

```
> minimize(f,x=-0.5..0.5,y=-0.5..0.5);
```

```
> minimize(f,x=-1/2..1/2,y=-1/2..1/2);
```

$$\frac{\frac{3}{4} + \frac{1}{432}(-15 + 3\sqrt{37})^2}{\sqrt{\frac{5}{12} + \frac{1}{6}\sqrt{37}}}$$

```
> simplify(%);
```

$$-\frac{1}{4} \frac{-49 + 5\sqrt{37}}{\sqrt{15 + 6\sqrt{37}}}$$

```
> evalf(%);
```

```
.6475020472
```

```
>
```

```
> f:=y*arcsin((2*x)/(1+x^2))/(3+y^4);
```

$$f := \frac{y \arcsin\left(2 \frac{x}{x^2 + 1}\right)}{3 + y^4}$$

```
> gr:=convert(grad(f,[x,y]),set);
```

$$gr := \left\{ \frac{y \left(2 \frac{1}{x^2+1} - \frac{4x^2}{(x^2+1)^2} \right)}{\sqrt{1 - \frac{4x^2}{(x^2+1)^2} (3+y^4)}}, \frac{\arcsin\left(2 \frac{x}{x^2+1}\right)}{3+y^4} - \frac{4y^4 \arcsin\left(2 \frac{x}{x^2+1}\right)}{(3+y^4)^2} \right\}$$

> gr:=simplify(convert(grad(f,[x,y]),set));

$$gr := \left\{ -2 \frac{y(x^2-1)}{(x^2+1)^2 \sqrt{\frac{(x^2-1)^2}{(x^2+1)^2} (3+y^4)}}, -3 \frac{\arcsin\left(2 \frac{x}{x^2+1}\right) (-1+y^4)}{(3+y^4)^2} \right\}$$

> s:=solve(gr,{x,y});

$$s := \{x=0, y=0\}$$

> he:=hessian(f,[x,y]):

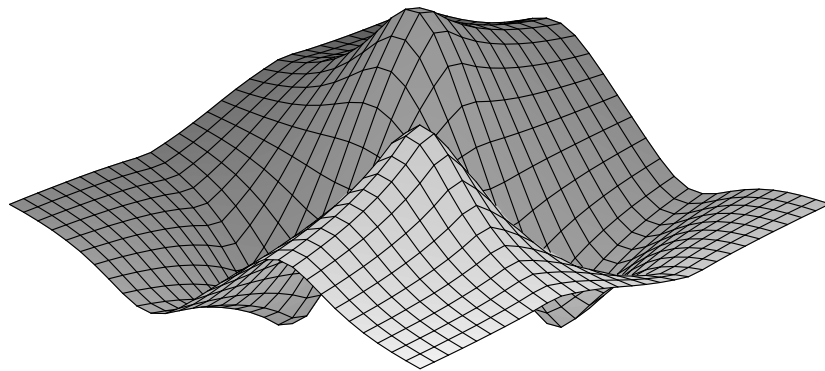
> subs(s,matrix(he));

$$\begin{bmatrix} 0 & \frac{2}{3} \\ \frac{2}{3} & 0 \end{bmatrix}$$

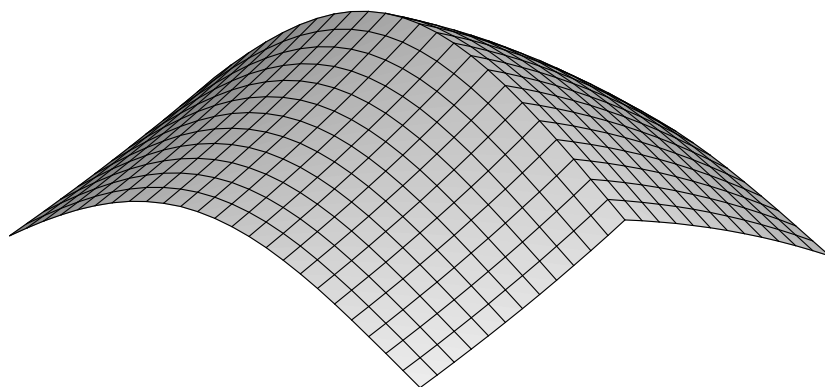
> eigenvals(%);

$$\frac{2}{3}, \frac{-2}{3}$$

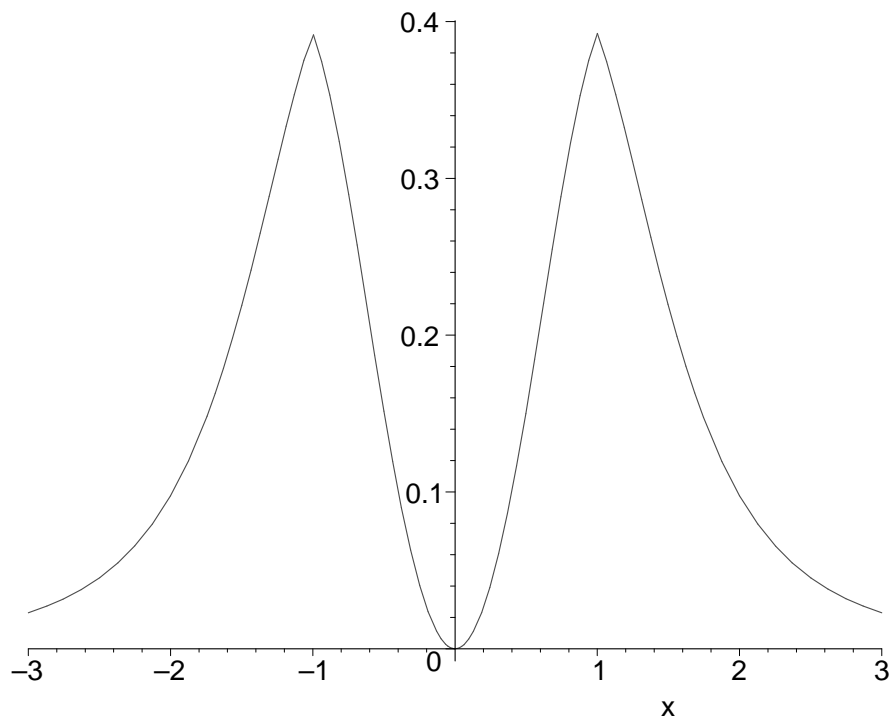
> plot3d(f,x=-3..3,y=-3..3,grid=[30,30]);



```
> plot3d(f,x=0.5..1.5,y=0.5..1.5);
```



```
> plot(subs(y=x,f),x=-3..3);
```



```
> subs({x=1,y=1},f);
```

$$\frac{1}{4} \arcsin(1)$$

```
> simplify(%);
```

$$\frac{1}{8} \pi$$

```
> subs({x=1,y=1},gr);
```

Error, numeric exception: division by zero

```
>
```

```
> A:=matrix(3,3,[0,1,2,3,4,3,2,1,0]);
```

$$A := \begin{bmatrix} 0 & 1 & 2 \\ 3 & 4 & 3 \\ 2 & 1 & 0 \end{bmatrix}$$

```
> det(A);
```

$$-4$$

```
> ei:=eigenvals(A);
```

$$ei := -2, 3 + \sqrt{7}, 3 - \sqrt{7}$$

```
> eigenvectors(A);
```

$$[3 + \sqrt{7}, 1, \{[1, 1 + \sqrt{7}, 1]\}], [3 - \sqrt{7}, 1, \{[1, 1 - \sqrt{7}, 1]\}], [-2, 1, \{[-1, 0, 1]\}]$$

```
> charpoly(A,x);
```

$$x^3 - 4x^2 - 10x + 4$$


```

> inverse(A);

$$\begin{bmatrix} \frac{3}{4} & -\frac{1}{2} & \frac{5}{4} \\ -\frac{3}{2} & 1 & -\frac{3}{2} \\ \frac{5}{4} & -\frac{1}{2} & \frac{3}{4} \end{bmatrix}$$

> multiply(A,A);

$$\begin{bmatrix} 7 & 6 & 3 \\ 18 & 22 & 18 \\ 3 & 6 & 7 \end{bmatrix}$$

> multiply(A,A,A);

$$\begin{bmatrix} 24 & 34 & 32 \\ 102 & 124 & 102 \\ 32 & 34 & 24 \end{bmatrix}$$

> b:=matrix(3,1,[1,3,c]);

$$b := \begin{bmatrix} 1 \\ 3 \\ c \end{bmatrix}$$

> linsolve(A,b);

$$\begin{bmatrix} \frac{3}{4} + \frac{5}{4}c \\ \frac{3}{2} - \frac{3}{2}c \\ -\frac{1}{4} + \frac{3}{4}c \end{bmatrix}$$

> A:=matrix(3,3,[0,1,2,3,a,3,2,1,0]);

$$A := \begin{bmatrix} 0 & 1 & 2 \\ 3 & a & 3 \\ 2 & 1 & 0 \end{bmatrix}$$

> det(A);

$$12 - 4a$$

> ei:=eigenvals(A);

```

$$ei := -2, \frac{1}{2}a + 1 + \frac{1}{2}\sqrt{a^2 - 4a + 28}, \frac{1}{2}a + 1 - \frac{1}{2}\sqrt{a^2 - 4a + 28}$$

> eigenvectors(A);

$$\left[\frac{1}{2}a + 1 + \frac{1}{2}\sqrt{a^2 - 4a + 28}, 1, \right.$$

$$\left. \left\{ \left[-\frac{1}{12}a + \frac{1}{6} + \frac{1}{12}\sqrt{a^2 - 4a + 28}, 1, -\frac{1}{12}a + \frac{1}{6} + \frac{1}{12}\sqrt{a^2 - 4a + 28} \right] \right\} \right], \left[\right.$$

$$\left. \frac{1}{2}a + 1 - \frac{1}{2}\sqrt{a^2 - 4a + 28}, 1, \right.$$

$$\left. \left\{ \left[-\frac{1}{12}a + \frac{1}{6} - \frac{1}{12}\sqrt{a^2 - 4a + 28}, 1, -\frac{1}{12}a + \frac{1}{6} - \frac{1}{12}\sqrt{a^2 - 4a + 28} \right] \right\} \right], [-2, 1, \{[-1, 0, 1]\}]$$

> solve(ei[1]=ei[2], a);

> solve(ei[1]=ei[3], a);

$$\frac{-1}{2}$$

> solve(ei[2]=ei[3], a);

$$2 + 2I\sqrt{6}, 2 - 2I\sqrt{6}$$

> eigenvectors(subs(a=-1/2, matrix(A)));

$$[-2, 2, \{[0, -2, 1], [1, -2, 0]\}], \left[\frac{7}{2}, 1, \left\{ \left[1, \frac{3}{2}, 1 \right] \right\} \right]$$

> charpoly(A, x);

$$x^3 - x^2 a - 10x - 12 + 4a$$

> inverse(A);

$$\begin{bmatrix} \frac{3}{4} \frac{1}{-3+a} & -\frac{1}{2} \frac{1}{-3+a} & \frac{1}{4} \frac{-3+2a}{-3+a} \\ -\frac{3}{2} \frac{1}{-3+a} & \frac{1}{-3+a} & -\frac{3}{2} \frac{1}{-3+a} \\ \frac{1}{4} \frac{-3+2a}{-3+a} & -\frac{1}{2} \frac{1}{-3+a} & \frac{3}{4} \frac{1}{-3+a} \end{bmatrix}$$

> multiply(A, A);

$$\begin{bmatrix} 7 & 2+a & 3 \\ 6+3a & 6+a^2 & 6+3a \\ 3 & 2+a & 7 \end{bmatrix}$$

> multiply(A, A, A);

$$\begin{bmatrix} 12+3a & 10+(2+a)a & 20+3a \\ 30+3a^2+6a & 12+6a+(6+a^2)a & 30+3a^2+6a \\ 20+3a & 10+(2+a)a & 12+3a \end{bmatrix}$$

> linsolve(A,b);

$$\begin{bmatrix} \frac{1}{4} \frac{-3c-3+2ca}{-3+a} \\ -\frac{3}{2} \frac{c-1}{-3+a} \\ \frac{1}{4} \frac{3c+2a-9}{-3+a} \end{bmatrix}$$

> subs(a=3,%);

Error, numeric exception: division by zero

> A0:=subs(a=3,matrix(A));

$$A0 := \begin{bmatrix} 0 & 1 & 2 \\ 3 & 3 & 3 \\ 2 & 1 & 0 \end{bmatrix}$$

> linsolve(A0,b);

> b0:=subs(c=1,matrix(b));

$$b0 := \begin{bmatrix} 1 \\ 3 \\ 1 \end{bmatrix}$$

> linsolve(A0,b0);

$$\begin{bmatrix} -t_{1_1} \\ 1-2-t_{1_1} \\ -t_{1_1} \end{bmatrix}$$

>
>