SOME RESEARCH PROBLEMS IN VALUE DISTRIBUTION AND COMPLEX DYNAMICS

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Most of the following problems were posed at the INTAS starting meeting in Lviv in October 2000.

Problem 1. Let f be an entire function of order ρ, with $\frac{1}{2} \leq \rho \leq \infty$. Denote by $n(r,a)$ the number of zeros of $f(z)-a$ in the disk given by $|z| \leq r$. It was shown in [1] that if $a, b \in \mathbb{C}$ are distinct, then

$$\limsup_{r\to\infty} \frac{n(r,a)+n(r,b)}{\log M(r,f)} \geq \frac{1}{\pi}. \tag{1}$$

An immediate consequence is that

$$\limsup_{r\to\infty} \frac{n(r,a)}{\log M(r,f)} \geq \frac{1}{2\pi} \tag{2}$$

for all $a \in \mathbb{C}$ with at most one exception.

Can the constant $1/2\pi$ in (2) be replaced by $1/\pi$? This would be best possible.

For $f = \exp$ and $a = 0$ we have equality in (1). Can the constant $1/\pi$ in (1) be improved for large ρ? In particular, does (1) hold with $1/\pi$ replaced by $2/\pi$ if $\rho = \infty$?

Problem 2. Let f be an entire function of order ρ, with $\frac{1}{2} \leq \rho < \infty$. It was shown by Govorov [8] and Petrenko [10] that

$$\liminf_{r\to\infty} \frac{\log M(r,f)}{T(r,f)} \leq \pi\rho. \tag{3}$$

For functions f satisfying

$$\Delta := \sum_{a\in\mathbb{C}} \delta(a,f) = 1$$

the inequality (3) can be improved to the equation

$$\lim_{r\to\infty} \frac{\log M(r,f)}{T(r,f)} = \pi.$$

Can (3) be improved if $0 < \Delta < 1$? In other words, find a good upper bound (or even better, the sharp upper bound) for

$$\liminf_{r\to\infty} \frac{\log M(r,f)}{T(r,f)}$$

in terms of ρ and Δ.

Problem 3. Let $f : \mathbb{C} \to \widehat{\mathbb{C}}$ be meromorphic of order ρ, with $\frac{1}{2} \leq \rho \leq \infty$. Let

$$A(r,f) = \frac{1}{\pi} \int_{|z|\leq r} \frac{|f'(z)|^2}{(1+|f(z)|^2)^2} dxdy,$$

$$b(\infty, f) = \liminf_{r\to\infty} \frac{\log M(r,f)}{A(r,f)}$$

and

$$b(a,f) = b\left(\infty, \frac{1}{f-a}\right)$$

for $a \in \mathbb{C}$. It was shown in [3] that $b(\infty, f) \leq \pi$. Eremenko [6] then proved that

$$B := \sum_{a\in\widehat{\mathbb{C}}} b(a,f) \leq 2\pi.$$

Eremenko [7] also proved that if $\rho < \infty$ and $B = 2\pi$, then $\rho = n/2$ for some $n \in \mathbb{N}$, $n \geq 2$, and $b(a,f) = 2\pi/n$ if $b(a,f) \neq 0$.

Is the hypothesis $\rho < \infty$ necessary here? In other words, does $B = 2\pi$ imply that $\rho < \infty$ (and thus that $\rho = n/2$)?

The first result of Eremenko can be considered as an analogue of the deficiency relation and the second one as an analogue of a conjecture of F. Nevanlinna, proved first by Drasin [4] and later also by Eremenko [5], concerning functions with deficieny sum 2.

A general problem would be to extend other results concerning the Nevanlinna deficiencies $\delta(a,f)$ or the Petrenko deviations $\beta(a,f)$ to the quantities $b(a,f)$.

Problem 4. Let $D_1, \ldots, D_5$ be Jordan domains on the Riemann sphere $\widehat{\mathbb{C}}$ with pairwise disjoint closures. The Ahlfors five islands theorem (or, more precisely, one version of it) says that if $f : \mathbb{C} \to \widehat{\mathbb{C}}$ is meromorphic and non-constant, then there exists $U \subset \mathbb{C}$ and $j \in \{1, \ldots, 5\}$ such that $f : U \to D_j$ is biholomorphic.

What are the exact hypothesis on the D_j that are needed? For example, is the result true if $D_j := \{z \in \mathbb{C} : j < |z| < j + \frac{1}{2}, |\arg z| < \pi\}$?

Problem 5. Let $f : \mathbb{C} \to \widehat{\mathbb{C}}$ be meromorphic. For a Jordan domain $D \subset \mathbb{C}$ we say that f has an island over D if there exists a bounded domain $U \subset \mathbb{C}$ such that $f : U \to D$ is a proper map.

Let now $D_0, D_1 \subset \mathbb{C}$ be Jordan domains with $0 \notin \overline{D_1}$. Suppose that f does not have an island over D_0 and that f' does not have an island over D_1. Does it follow that f is constant?

If the answer is yes, this would generalize a theorem of Hayman [9] which says that if $f : \mathbb{C} \to \widehat{\mathbb{C}}$ is meromorphic with $f(z) \neq 0$ and $f'(z) \neq 1$ for all $z \in \mathbb{C}$, then f is constant.

It was shown in [2] that there exists $\varepsilon > 0$ such that if $f(z) \neq 0$ for $z \in \mathbb{C}$ and f' has no island over $\{z \in \mathbb{C} : |z - 1| < \varepsilon\}$, then f is constant.

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