# FATOU-JULIA THEORY FOR NON-UNIFORMLY QUASIREGULAR MAPS

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Abstract. Many results of the Fatou-Julia iteration theory of rational functions extend to uniformly quasiregular maps in higher dimensions. We obtain results of this type for certain classes of quasiregular maps which are not uniformly quasiregular.

## 1. Introduction and main results

Quasiregular maps are a natural generalization of holomorphic maps to higher dimensions. It is the purpose of this paper to show that certain results of holomorphic dynamics have analogs for quasiregular maps. We will recall the definition and basic properties of quasiregular maps in section 2, defining in particular terms like the dilatation $K(f)$ and the inner dilatation $K_I(f)$ of a quasiregular map $f$ that are used in the following.

An important result about quasiregular maps is Rickman's [22, 23] analog of Picard's theorem. He showed that there exists a constant $q = q(n, K)$ such that if $a_1, \dots, a_q \in \mathbb{R}^n$ are distinct and $f : \mathbb{R}^n \to \mathbb{R}^n \setminus \{a_1, \dots, a_q\}$ is $K$-quasiregular, then $f$ is constant. Note that Picard's theorem says that $q(2, 1) = 2$.

Miniowitz [19] used an extension of the Zalcman lemma [36] to quasiregular maps to obtain an analog of Montel's theorem from Rickman's result. Given the central role of Montel's theorem in holomorphic dynamics, it is seems clear that Miniowitz's theorem will be important in quasiregular dynamics. However, in order to apply this result to the family $\{f^j\}$ of iterates a quasiregular map $f$, one has to assume that all $f^j$ are $K$-quasiregular with the same $K$. Quasiregular maps with this property are called *uniformly quasiregular*. For uniformly quasiregular self-maps of the one point compactification $\overline{\mathbb{R}^n} := \mathbb{R}^n \cup \{\infty\}$ of $\mathbb{R}^n$ an iteration theory in the spirit of Fatou and Julia has been developed by Hinkkanen, Martin, Mayer and others [12, 14, 17]; see [3, Section 4], [13, Chapter 21] and [26, Chapter 4] for surveys.

As in the classical case of rational maps, the Julia set $J(f)$ of a uniformly quasiregular map $f : \overline{\mathbb{R}^n} \to \overline{\mathbb{R}^n}$ is defined as the set of all points where the family of iterates fails to be normal. Assuming that the degree of $f$ is at least 2 one finds that $J(f)$ is perfect; in particular, $J(f) \neq \emptyset$. Here the degree $\deg(f)$ of a (not necessarily uniformly) quasiregular map $f : \overline{\mathbb{R}^n} \to \overline{\mathbb{R}^n}$ is defined as the

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maximal cardinality of the preimage of a point; that is,

$$\deg(f) := \max_{x\in\overline{\mathbb{R}^n}} \operatorname{card} f^{-1}(x),$$

where $\operatorname{card} A$ denotes the cardinality of a set $A$.

For $x \in \overline{\mathbb{R}^n}$ we define the forward orbit $O^+(x) := \{f^j(x) : j \in \mathbb{N}\}$ and for $X \subset \overline{\mathbb{R}^n}$ we put $O^+(X) := \bigcup_{x\in X} O^+(x)$. One direct consequence of Miniowitz's theorem is the so-called expansion property which says that if $U$ is an open set intersecting the Julia set, then $\overline{\mathbb{R}^n} \setminus O^+(U)$ is finite. In fact, this set contains at most $q(n,K)$ points, provided $K(f^j) \le K$ for all $j \in \mathbb{N}$.

We refer to the papers mentioned above – and the references cited therein – for further results about the dynamics of uniformly quasiregular maps.

Sun and Yang [29, 30, 31] showed that in dimension 2 some results of the Fatou-Julia theory still hold even for non-uniformly quasiregular maps, provided the degree exceeds the dilatation. However, the definition of the Julia set via non-normality is not adequate here. Instead Sun and Yang used the expansion property to define the Julia set. They thus defined the Julia set $J(f)$ of a quasiregular self-map $f$ of the Riemann sphere $\overline{\mathbb{C}}$ as the set of all $z \in \overline{\mathbb{C}}$ such that $\overline{\mathbb{C}} \setminus O^+(U)$ contains at most two points, for every neighborhood $U$ of $z$. They showed that if $\deg(f) > K(f)$, then $J(f) \ne \emptyset$, and many results of the Fatou-Julia theory hold. For an exposition of their results we refer to [3, Section 5].

There have been only a few papers concerned with the the dynamics of non-uniformly quasiregular maps in higher dimensions. In [4, 5, 6] certain quasiregular maps $f : \mathbb{R}^n \to \mathbb{R}^n$ with an essential singularity at infinity were considered. Such maps can be thought of as analogs of transcendental entire functions. In contrast, a quasiregular map $f : \mathbb{R}^n \to \mathbb{R}^n$ is said to be of *polynomial type* if $\lim_{x\to\infty} f(x) = \infty$. Such a map $f$ extends to a quasiregular self-map of $\overline{\mathbb{R}^n}$ by putting $f(\infty) = \infty$. The dynamics of such maps were studied by Fletcher and Nicks [8] who proved that if $\deg(f) > K_I(f)$, then the boundary of the *escaping set* $I(f) := \{x \in \mathbb{R}^n : f^j(x) \to \infty\}$ has many properties usually associated with the Julia set. Note that $J(f) = \partial I(f)$ for non-linear polynomials $f : \mathbb{C} \to \mathbb{C}$, as well as transcendental entire functions [7].

We shall be concerned with quasiregular self-maps of $\overline{\mathbb{R}^n}$ which need not be of polynomial type. Such maps can be considered as analogs of rational functions. In order to state our first result, we need to introduce sets of capacity zero; cf. [24, section II.10]. For an open set $G \subset \mathbb{R}^n$ and a non-empty compact subset $C$ of $G$ the pair $(G, C)$ is called a *condenser* and its *capacity* $\operatorname{cap}(G, C)$ is defined by

$$\operatorname{cap}(G, C) := \inf_u \int_G |\nabla u|^n \, dm,$$

where the infimum is taken over all non-negative functions $u \in C_0^\infty(G)$ satisfying $u(x) \ge 1$ for all $x \in C$. (Here $C_0^\infty(G)$ may be replaced by the Sobolev space $W^1_{n,\mathrm{loc}}(G)$, which also appears in the definition of quasiregularity; cf. section 2.)

It turns out [24, Lemma III.2.2] that if $\operatorname{cap}(G, C) = 0$ for some bounded open set $G$ containing $C$, then $\operatorname{cap}(G', C) = 0$ for every bounded open set $G'$ containing $C$. In this case we say that $C$ is of *capacity zero* and denote this by

$\operatorname{cap} C = 0$. Otherwise we say that $C$ has *positive capacity* and write $\operatorname{cap} C > 0$. Note that this does not mean that $\operatorname{cap} C$ is a positive number. (The capacity is defined for condensers, not for sets.) However, we mention that Vuorinen [32] has introduced a set function $c$ satisfying $c(C) > 0$ if and only if $\operatorname{cap} C > 0$. Möbius transformations preserve the capacity of a condenser and hence preserve sets of capacity zero, leading to an obvious extension of the definition to subsets of $\overline{\mathbb{R}^n}$; see [21, Section 1.3] for the definition and a discussion of Möbius transformations.

We mention that sets of capacity zero are totally disconnected [24, Corollary III.2.5] and in fact of Hausdorff dimension zero [24, Corollary VII.1.15]; see also Lemma 10.1 below for a stronger statement involving Hausdorff measure. On the other hand, a finite set has capacity zero.

**Theorem 1.1.** *Let $f : \overline{\mathbb{R}^n} \to \overline{\mathbb{R}^n}$ be quasiregular. Suppose that* $\deg(f) > K_I(f)$. *Then there exists $x \in \overline{\mathbb{R}^n}$ such that*

$$\operatorname{cap}\big(\overline{\mathbb{R}^n} \setminus O^+(U)\big) = 0 \tag{1.1}$$

*for every neighborhood $U$ of $x$.*

As in [8, 29] the winding map (cf. [24, Section I.3.1]) shows that the hypothesis that $\deg(f) > K_I(f)$ cannot be weakened to $\deg(f) \geq K_I(f)$.

We note that if $f$ is uniformly quasiregular and $\deg(f) \geq 2$, then the hypothesis of Theorem 1.1 is satisfied for some iterate of $f$. The hypothesis of Theorem 1.1 and subsequent theorems could be weakened to $\deg(f^p) > K_I(f^p)$ for some $p \in \mathbb{N}$ in order to cover all uniformly quasiregular maps, but for simplicity we restrict ourselves to the case $p = 1$.

We mention that the composition of a uniformly quasiregular map with a Möbius transformation need not be uniformly quasiregular. In contrast, the hypothesis of Theorem 1.1 is preserved under compositions with Möbius transformations. This yields many examples of quasiregular maps satisfying the hypothesis of Theorem 1.1 which are not uniformly quasiregular.

Following Sun and Yang we define the Julia set as follows.

**Definition 1.1.** *Let $f : \overline{\mathbb{R}^n} \to \overline{\mathbb{R}^n}$ be quasiregular. Then the set of all $x \in \overline{\mathbb{R}^n}$ such that* (1.1) *holds for every neighborhood $U$ of $x$ is called the* Julia set *of $f$ and denoted by $J(f)$.*

Theorem 1.1 says that $J(f) \neq \emptyset$ if $\deg(f) > K_I(f)$. As in the case of rational functions it is easy to see that $J(f)$ is closed and completely invariant; cf. [2, Theorem 3.2.4], [18, Lemma 4.3] or [28, Section 25]. Here a set $A$ is called *completely invariant* (under $f$) if $x \in A$ implies that $f(x) \in A$, and vice versa. It follows that $J(f)$ has empty interior unless $J(f) = \overline{\mathbb{R}^n}$; cf. [2, Theorem 4.2.3], [18, Corollary 4.11] or [28, Section 30].

Definition 1.1 is justified by the following result.

**Theorem 1.2.** *For a uniformly quasiregular map $f : \overline{\mathbb{R}^n} \to \overline{\mathbb{R}^n}$ the definition of $J(f)$ using non-normality coincides with the one given in Definition 1.1.*

A point $\xi \in \overline{\mathbb{R}^n}$ is called *periodic* if there exists $p \in \mathbb{N}$ such that $f^p(\xi) = \xi$. The smallest $p$ with this property is called the *period* of $\xi$. We denote by $\chi$ the chordal metric on $\overline{\mathbb{R}^n}$, obtained via stereographic projection from the unit sphere in $\mathbb{R}^{n+1}$. Following Sun and Yang [31, Definition 4] we say that a periodic point $\xi$ of period $p$ is *attracting* if there exists $c \in (0,1)$ and a neighborhood $U$ of $\xi$ such that $\chi(f^p(z),\xi) < c\,\chi(z,\xi)$ for all $z \in U$. Similarly we say that $\xi$ is *repelling* if $\chi(f^p(z),\xi) > c\,\chi(z,\xi)$ for some $c > 1$ and all $z$ in some neighborhood of $\xi$.

We note that other definitions of attracting and repelling have been used for uniformly quasiregular maps (cf. [12] for a discussion), but all definitions have in common that an attracting periodic point of period $p$ has a neighborhood where the iterates of $f^p$ converge uniformly to it. For an attracting periodic point $\xi$ of period $p$ the set

$$A(\xi) := \{x \in \overline{\mathbb{R}^n} : \lim_{j\to\infty} f^{pj}(x) = \xi\},$$

called the *attracting basin* of $\xi$, thus contains a neighborhood of $\xi$.

**Theorem 1.3.** *Let $f : \overline{\mathbb{R}^n} \to \overline{\mathbb{R}^n}$ be quasiregular with* $\deg(f) > K_I(f)$. *If $\xi$ is an attracting periodic point of $f$, then $J(f) \cap A(\xi) = \emptyset$ and $J(f) \subset \partial A(\xi)$.*

For rational functions and, more generally, uniformly quasiregular maps we have $J(f) = \partial A(\xi)$; see [18, Corollary 4.12]. As shown in [3, Example 5.3], this need not be the case in the present setting.

For a map $f : \overline{\mathbb{R}^n} \to \overline{\mathbb{R}^n}$ the *exceptional set* $E(f)$ is defined as the set of all $x \in \overline{\mathbb{R}^n}$ for which the backward orbit $O^-(x) := \bigcup_{j=1}^{\infty} f^{-j}(x)$ is finite.

**Theorem 1.4.** *Let $f : \overline{\mathbb{R}^n} \to \overline{\mathbb{R}^n}$ be quasiregular with* $\deg(f) > K_I(f)$. *Then $E(f)$ is finite and consists of attracting periodic points. In particular, $E(f)$ does not intersect $J(f)$.*

This result is standard for rational functions; see [2, Section 4.1], [18, Lemma 4.9] or [28, Section 31]. For uniformly quasiregular maps it can be found in, e.g., [26, pp. 64–65].

Quasiregular maps are Hölder continuous; cf. section 7. For the analogs of some further key results of complex dynamics we require the stronger hypothesis of Lipschitz continuity. This condition is satisfied for many examples of uniformly quasiregular maps. We also note that uniformly quasiregular maps are Lipschitz continuous at fixed points [12, Lemma 4.1].

**Theorem 1.5.** *Let $f : \overline{\mathbb{R}^n} \to \overline{\mathbb{R}^n}$ be quasiregular with* $\deg(f) > K_I(f)$. *Suppose that $f$ is Lipschitz continuous. If $U$ is an open set intersecting $J(f)$, then $O^+(U) \supset \overline{\mathbb{R}^n} \setminus E(f)$ and $O^+(U \cap J(f)) = J(f)$.*

**Theorem 1.6.** *Let $f$ be as in Theorem 1.5. Then $J(f) = \overline{O^-(x)}$ for every $x \in J(f)$ and $J(f) \subset \overline{O^-(x)}$ for every $x \in \overline{\mathbb{R}^n} \setminus E(f)$.*

Theorems 1.5 and 1.6 are well-known for rational functions; see [2, Theorems 4.2.5 and 4.2.7], [18, Theorem 4.10 and Corollary 4.13] or [28, Sections 28 and 32]. For uniformly quasiregular maps these results are – as already mentioned – consequences of Miniowitz's theorem and can be found in, e.g., [12, Section 3].

We denote the Hausdorff dimension of a subset $A$ of $\mathbb{R}^n$ by $\dim A$.

**Theorem 1.7.** *Let $f$ be as in Theorem 1.5. Then* $\dim J(f) > 0$.

For rational functions Theorem 1.7 is due to Garber; see [10], [2, Section 10.3] or [28, Section 168]. For uniformly quasiregular maps it was recently proved by Fletcher and Nicks [9].

We conjecture that the hypothesis that $f$ is Lipschitz continuous can be omitted in Theorems 1.5 and 1.6, but not in Theorem 1.7. However, we conjecture that without this hypothesis we still have $\operatorname{cap} J(f) > 0$. We prove that this is the case under an additional assumption involving the *branch set* $B_f$ which is defined as the set of all points where $f$ is not locally injective; cf. section 2.

**Theorem 1.8.** *Let $f : \overline{\mathbb{R}^n} \to \overline{\mathbb{R}^n}$ be quasiregular with* $\deg(f) > K_I(f)$. *Suppose that $J(f) \cap B_f = \emptyset$. Then* $\operatorname{cap} J(f) > 0$.

This paper is organized as follows. In section 2 we recall the definition of quasiregular maps and in section 3 we state some results about averages of counting functions which play a key role in the proof of Theorem 1.1. These results will be proved in section 5, using some lemmas about the capacities of condensers and the moduli of path families given before in section 4. Theorems 1.1–1.3 are then proved in section 6. An important ingredient in various arguments are results about the local distortion of quasiregular maps. We discuss and prove such results in section 7 and use them in section 8 to prove Theorem 1.4. In section 9 we obtain some results about the Hausdorff measure of invariant sets and use them in section 10 to prove Theorems 1.5–1.8. In section 11 we give some evidence for the conjecture made above that Theorems 1.5 and 1.6 hold without the hypothesis of Lipschitz continuity. We also show that the conclusion of these theorems holds under some different hypothesis.

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## 2. Quasiregular maps

We denote the (Euclidean) norm of a point $x \in \mathbb{R}^n$ by $|x|$. For $a \in \mathbb{R}^n$ and $r > 0$ let $B(a,r) := \{x \in \mathbb{R}^n : |x-a| < r\}$ be the open ball, $\overline{B}(a,r)$ the closed ball and $S(a,r) := \partial B(a,r)$ the sphere of radius $r$ centered at $a$. We write $B(r)$, $\overline{B}(r)$ and $S(r)$ instead of $B(0,r)$, $\overline{B}(0,r)$ and $S(0,r)$. Sometimes we will emphasize the dimension by writing $B^n(a,r)$, $S^{n-1}(a,r) = \partial B^n(a,r)$, etc. With the stereographic projection $\pi : S^n(1) \to \overline{\mathbb{R}^n}$ the chordal metric $\chi$ already mentioned is given by $\chi(x,y) = |\pi^{-1}(x) - \pi^{-1}(y)|$. (Instead of the chordal metric, one could also use the spherical metric.) Balls with respect to the chordal metric are denoted by a subscript $\chi$; that is, $B_\chi(a,r) := \{x \in \overline{\mathbb{R}^n} : \chi(x,a) < r\}$.

We recall the definition of quasiregularity; see Rickman's monograph [24] for more details. Let $n \geq 2$ and let $\Omega \subset \mathbb{R}^n$ be a domain. For $1 \leq p < \infty$ the Sobolev space $W^1_{p,\mathrm{loc}}(\Omega)$ is defined as the set of functions $f = (f_1, \dots, f_n) : \Omega \to \mathbb{R}^n$ for which all first order weak partial derivatives $\partial_k f_j$ exist and are locally in $L^p$. It turns out that a continuous map $f$ is in $W^1_{p,\mathrm{loc}}(\Omega)$ if and only if all $f_j$ are absolutely continuous on almost all lines parallel to the coordinate axes, with all partial derivatives locally $L^p$-integrable. For us only the case $p = n$ will be of interest.

A continuous map $f \in W^1_{n,\mathrm{loc}}(\Omega)$ is called *quasiregular* if there exists a constant $K_O \geq 1$ such that

$$|Df(x)|^n \leq K_O J_f(x) \quad \text{a.e.}, \tag{2.1}$$

where $Df(x)$ denotes the derivative,

$$|Df(x)| := \sup_{|h|=1} |Df(x)(h)|$$

its norm, and $J_f(x)$ the Jacobian determinant. With

$$\ell(Df(x)) := \inf_{|h|=1} |Df(x)(h)|$$

the condition that (2.1) holds for some $K_O \geq 1$ is equivalent to the condition that

$$J_f(x) \leq K_I \ell(Df(x)) \quad \text{a.e.}, \tag{2.2}$$

for some $K_I \geq 1$. The smallest constants $K_O$ and $K_I$ for which (2.1) and (2.2) hold are called the *outer and inner dilatation* of $f$ and denoted by $K_O(f)$ and $K_I(f)$. Moreover, $K(f) := \max\{K_I(f), K_O(f)\}$ is called the (maximal) *dilatation* of $f$. We say that $f$ is $K$-*quasiregular* if $K(f) \leq K$.

If $f$ and $g$ are quasiregular, with $f$ defined in the range of $g$, then $f \circ g$ is also quasiregular and [24, Theorem II.6.8]

$$K_I(f \circ g) \leq K_I(f) K_I(g) \quad \text{and} \quad K_O(f \circ g) \leq K_O(f) K_O(g) \tag{2.3}$$

so that $K(f \circ g) \leq K(f)K(g)$.

As already mentioned, many properties of holomorphic functions carry over to quasiregular maps. Here we only note that non-constant quasiregular maps are open and discrete. We refer to the monographs [21, 24] for a detailed treatment of quasiregular maps.

The *local index* $i(x, f)$ of a quasiregular map $f : \Omega \to \mathbb{R}^n$ at a point $x \in \Omega$ is defined by

$$i(x, f) := \inf_U \sup_{y \in \mathbb{R}^n} \operatorname{card}\left(f^{-1}(y) \cap U\right),$$

where the infimum is taken over all neighborhoods $U \subset \Omega$ of $x$. We thus have $i(x, f) = 1$ if and only if $f$ is injective in a neighborhood of $x$. The *branch set* $B_f$ already mentioned in the introduction consists of all $x \in \Omega$ for which $i(x, f) \geq 2$.

Quasiregularity can be defined more generally for maps between Riemannian manifolds. Here we consider only the case that the domain or range are equal to (or contained in) $\overline{\mathbb{R}^n}$. It turns out that for a domain $\Omega \subset \overline{\mathbb{R}^n}$ a non-constant continuous map $f : \Omega \to \overline{\mathbb{R}^n}$ is quasiregular if $f^{-1}(\infty)$ is discrete and if $f$ is quasiregular in $\Omega \setminus (f^{-1}(\infty) \cup \{\infty\})$.

## 3. Averages of counting functions

For a quasiregular map $f : \Omega \to \overline{\mathbb{R}^n}$, a compact subset $E$ of $\Omega$ and $y \in \overline{\mathbb{R}^n}$ we denote by $n(E, y)$ the number of $y$-points of $f$ in $E$, counted according to multiplicity. Thus

$$n(E, y) = \sum_{x \in f^{-1}(y) \cap E} i(x, f).$$

We will consider the average value of $n(E,y)$ over a sphere $S(z,t)$ and denote this by $\nu(E,S(z,t))$. Denoting the normalized $d$-dimensional Hausdorff measure by $H^d$ and putting $\omega_d = H^d(S^d(1))$ for $d \in \mathbb{N}$ we thus have

$$\nu(E,S(z,t)) = \frac{1}{\omega_{n-1}t^{n-1}} \int_{S(z,t)} n(E,y)dH^{n-1}(y).$$

We will mainly be concerned with the case that $E = \overline{B}(r)$. In this case we use the notation $n(r,y)$ and $\nu(r,S(z,t))$ instead of $n\big(\overline{B}(r),y\big)$ and $\nu\big(\overline{B}(r),S(z,t)\big)$.

The following result is obtained by careful inspection and suitable modification of a result of Mattila and Rickman [16, Lemma 3.3]. We shall give the proof in section 5.

**Theorem 3.1.** *There exists a constant $C$ depending only on the dimension $n$ such that if $F \subset B^n(z,t/2)$ is a compact set of positive capacity, $\theta > 1$ and $f : B^n(\theta r) \to \overline{\mathbb{R}^n} \setminus F$ is quasiregular, then*

$$\nu(r,S(z,t)) \le C \frac{K_I(f)}{(\log\theta)^{n-1}\operatorname{cap}(B^n(z,t),F)}. \tag{3.1}$$

The average of $n(E,y)$ over $\overline{\mathbb{R}^n}$ is denoted by $A(E)$. Identifying $\overline{\mathbb{R}^n}$ with $S^n(1)$ we thus have

$$A(E) = \frac{1}{\omega_n} \int_{S^n(1)} n(E,y)dH^n(y).$$

Similarly as before we write $A(r)$ instead of $A\big(\overline{B}(r)\big)$, and sometimes we include the map $f$ by writing $A(r,f)$.

It is shown in [24, Lemma IV.1.7] that $\nu(r,S(z,t))$ and $A(r)$ are comparable in the following sense.

**Lemma 3.1.** *There exists a constant $Q$ depending only on the dimension $n$ such that if $Y$ is an $(n-1)$-sphere of spherical radius $u \le \pi/4$, if $R > \theta r > r > 0$ and if $f : B^n(R) \to \overline{\mathbb{R}^n}$ is quasiregular, then*

$$\nu(r/\theta,Y) - Q\frac{K_I(f)|\log u|^{n-1}}{(\log\theta)^{n-1}} \le A(r) \le \nu(\theta r,Y) + Q\frac{K_I(f)|\log u|^{n-1}}{(\log\theta)^{n-1}}.$$

Noting that given a set $F$ of positive capacity and $t > 0$ we can find a subset of $F$ which has positive capacity and is contained in a ball of radius $t/2$, we obtain the following result from Theorem 3.1 and Lemma 3.1.

**Theorem 3.2.** *Let $F \subset \overline{\mathbb{R}^n}$ be a set of positive capacity and let $\theta > 1$. Then there exists a constant $C$ depending only on $n$, $F$ and $\theta$ such that if $f : B^n(\theta r) \to \overline{\mathbb{R}^n} \setminus F$ is quasiregular, then $A(r,f) \le C\, K_I(f)$.*

Clearly, it is irrelevant here that the balls considered are centered at 0 so that if $a \in \mathbb{R}^n$ and $f : B(a,\theta r) \to \mathbb{R}^n \setminus F$ is quasiregular, then $A\big(\overline{B}(a,r),f\big) \le C\, K_I(f)$. Similarly, we may consider balls with respect to the chordal metric and obtain $A\big(\overline{B}_\chi(a,r),f\big) \le C\, K_I(f)$ if $a \in \overline{\mathbb{R}^n}$, $0 < r < \theta r < 2$ and $f : B_\chi(a,\theta r) \to \overline{\mathbb{R}^n} \setminus F$ is quasiregular.

## 4. Capacity and the modulus of a path family

The modulus of a path family is a major tool in the study of quasiregular maps. We review this concept only briefly; see [24, Chapter II] and [33, Chapter 2] for more details. Let $\Gamma$ be a family of paths in $\mathbb{R}^n$. We say that a non-negative Borel function $\rho : \mathbb{R}^n \to \mathbb{R} \cup \{\infty\}$ is *admissible* if $\int_\gamma \rho \, ds \geq 1$ for all locally rectifiable paths $\gamma \in \Gamma$ and denote by $\mathcal{F}(\Gamma)$ the family of all admissible Borel functions. Then

$$M(\Gamma) := \inf_{\rho \in \mathcal{F}(\Gamma)} \int_{\mathbb{R}^n} \rho^n \, dm$$

is called the *modulus* of $\Gamma$. For the extension to families of paths in $\overline{\mathbb{R}^n}$ we refer to [33, pp. 53–54].

For a domain $G \subset \overline{\mathbb{R}^n}$ and sets $E, F \subset \overline{G}$ we denote by $\Delta(E, F; G)$ the family of all paths which have one endpoint in $E$, one endpoint in $F$ and which are in $G$ otherwise. The connection with capacity is given by the following result [24, Proposition II.10.2].

**Lemma 4.1.** *Let $G \subset \mathbb{R}^n$ be open and $C \subset G$ compact. Then*

$$\operatorname{cap}(G, C) = M(\Delta(C, \partial G; G)).$$

As an example we mention that for $0 < r < s$ we have [24, p. 28]

$$\operatorname{cap}\big(B(s), \overline{B}(r)\big) = M\big(\Delta\big(S(r), S(s); B(s) \backslash \overline{B}(r)\big)\big) = \omega_{n-1} \left(\log \frac{s}{r}\right)^{1-n}. \tag{4.1}$$

For two path families $\Gamma_1$ and $\Gamma_2$ we write $\Gamma_1 < \Gamma_2$ if every $\gamma \in \Gamma_2$ has a subpath belonging to $\Gamma_1$. As Ahlfors [1, p. 54] puts it: $\Gamma_2$ has fewer and longer arcs. The following lemma [24, p. 26] follows directly from the definition.

**Lemma 4.2.** *If $\Gamma_1 < \Gamma_2$, then $M(\Gamma_1) \geq M(\Gamma_2)$.*

We note that it follows from the definition of capacity, or from Lemma 4.1 and Lemma 4.2, that

$$\operatorname{cap}(C, G) \geq \operatorname{cap}(C, G') \quad \text{if } G \subset G'. \tag{4.2}$$

The next lemma is known as Väisälä's inequality [24, Theorem II.9.1].

**Lemma 4.3.** *Let $f$ be quasiregular in a domain $\Omega \subset \overline{\mathbb{R}^n}$, let $\Gamma^*$ be a path family in $\Omega$ and let $\Gamma$ be a path family in $\overline{\mathbb{R}^n}$. Suppose that there exists $m \in \mathbb{N}$ such that for every path $\beta : I \to \overline{\mathbb{R}^n}$ in $\Gamma$ there are paths $\alpha_1, \ldots, \alpha_m$ in $\Gamma^*$ such that $f \circ \alpha_j \subset \beta$ for all $j$ and such that for every $x \in \Omega$ and $t \in I$ the equality $\alpha_j(t) = x$ holds for at most $i(x, f)$ indices $j$. Then*

$$M(\Gamma) \leq \frac{K_I(f)}{m} M(\Gamma^*).$$

The following result [24, Theorem II.10.11] is a consequence of Lemma 4.3.

**Lemma 4.4.** *Let $f : \Omega \to \mathbb{R}^n$ be quasiregular, let $(G, C)$ be a condenser in $\Omega$ and put $m := \inf_{y \in f(C)} n(C, y)$. Then*

$$\operatorname{cap}(f(G), f(C)) \leq \frac{K_I(f)}{m} \operatorname{cap}(G, C).$$

As mentioned, the proof of Theorem 3.1 follows the arguments of Mattila and Rickman [16]. The following lemma is taken from their paper [16, Lemma 3.2].

**Lemma 4.5.** *Let $n \geq 2$ and $0 < u < v < \infty$. For $F_1 \subset \overline{B^n}(u)$ and $F_2 \subset S^{n-1}(v)$ define the path families $\Sigma_{12} := \Delta(F_1, F_2; B(v))$, $\Sigma_1 := \Delta(F_1, S(v); B(v))$ and $\Sigma_2 := \Delta(F_2, S(u); B(v) \backslash \overline{B}(u))$. Then*

$$M(\Sigma_{12}) \geq 3^{-n} \min\{M(\Sigma_1), M(\Sigma_2), c_n \log(v/u)\},$$

*where $c_n$ depends only on $n$.*

Note that with the terminology of Lemma 4.5 we have $M(\Sigma_1) = \operatorname{cap}(B(v), F_1)$ by Lemma 4.1.

The next lemma is implicit in the proof of [16, Lemma 3.3], but for completeness we include the proof.

**Lemma 4.6.** *For $n \in \mathbb{N}$ there exist positive constants $\alpha$ and $\beta$ such that if $r > 0$ and $A \subset S^{n-1}(r)$ is compact, then*

$$M\big(\Delta\big(S(r/2), A; B(r) \backslash \overline{B}(r/2)\big)\big) \geq \alpha \left( \log \left( \frac{\beta r^{n-1}}{H^{n-1}(A)} \right) \right)^{1-n}.$$

Here the right hand side is understood to be 0 if $H^{n-1}(A) = 0$. *Proof of Lemma 4.6.* By a result of Gehring [11, Lemma 1] we have

$$M\big(\Delta\big(S(r/2), A; B(r) \backslash \overline{B}(r/2)\big)\big) = \frac{1}{2} M\big(\Delta\big(S(r/2) \cup S(2r), A; B(2r) \backslash \overline{B}(r/2)\big)\big).$$

Thus

$$\begin{aligned} M\big(\Delta\big(S(r/2), A; B(r) \backslash \overline{B}(r/2)\big)\big) &= \frac{1}{2} \operatorname{cap}\big(B(2r) \backslash \overline{B}(r/2), A\big) \\ &\geq \frac{1}{2} \operatorname{cap}(B(2r), A) \end{aligned}$$

by Lemma 4.1 and (4.2).

We may assume that $H^{n-1}(A) > 0$ and denote by $A^*$ the spherical symmetrization of $A$; that is, using the notation $e_k$ for the $k$-th unit vector we put $A^* = S(r) \cap \overline{B}(re_n, s)$, where $s$ is chosen such that $H^{n-1}(A) = H^{n-1}(A^*)$. By a result of Sarvas [25] we have

$$\operatorname{cap}(B(2r), A) \geq \operatorname{cap}(B(2r), A^*).$$

Combining the last two estimates we obtain

$$M\big(\Delta\big(S(r/2), A; B(r) \backslash \overline{B}(r/2)\big)\big) \geq \frac{1}{2} \operatorname{cap}(B(2r), A^*). \tag{4.3}$$

We note that the modulus is invariant under translations. With $T(x) = x - re_n$ we thus have

$$\operatorname{cap}(B(2r), A^*) = \operatorname{cap}(T(B(2r)), T(A^*)) \geq \operatorname{cap}(B(3r), T(A^*)) \tag{4.4}$$

by (4.2). Now there exists $c > 0$ such that

$$(\operatorname{diam} A^*)^{n-1} \geq c H^{n-1}(A^*) = c H^{n-1}(A),$$

where $\operatorname{diam} A^*$ denotes the diameter of $A^*$. Thus there exists $a \in A^*$ with

$$|T(a)| = |a - re_n| \geq \frac{1}{2}\left(cH^{n-1}(A)\right)^{1/(n-1)}.$$

Since $T(A^*)$ is connected and $0 \in T(A^*)$, the extremality of the Grötzsch condenser

$$E_G(t) := (B^n(1), [0, te_1])$$

yields [24, Lemma III.1.9]

$$\operatorname{cap}(B(3r), T(A^*)) \geq \operatorname{cap} E_G\left(\frac{|T(a)|}{3r}\right).$$

Combining this with the estimate [24, Lemma III.1.2]

$$\operatorname{cap} E_G(t) \geq \omega_{n-1}\left(\log\frac{\lambda_n}{t}\right)^{1-n},$$

where $\lambda_n$ depends only on $n$, we obtain

$$\begin{aligned}\operatorname{cap}(B(3r), T(A^*)) &\geq \omega_{n-1}\left(\log\frac{3\lambda_n r}{|T(a)|}\right)^{1-n} \\ &\geq \omega_{n-1}\left(\log\frac{6\lambda_n r}{\left(cH^{n-1}(A)\right)^{1/(n-1)}}\right)^{1-n} \qquad (4.5)\\ &= 2\alpha\left(\log\left(\frac{\beta r^{n-1}}{H^{n-1}(A)}\right)\right)^{1-n}\end{aligned}$$

for suitable constants $\alpha$ and $\beta$ depending only on $n$. The conclusion follows from (4.3), (4.4) and (4.5). □

We conclude this section with the following lemma already mentioned in the introduction; see [24, Corollary VII.1.15].

**Lemma 4.7.** *Let $X \subset \mathbb{R}^n$ be compact. If $\dim X > 0$, then $\operatorname{cap} X > 0$.*

A strengthened form of Lemma 4.7 is given by Lemma 10.1 below.

## 5. Proof of Theorem 3.1

Without loss of generality we may assume that $z = 0$. For $k \in \mathbb{N}$ let

$$A_k := \{y \in S(t) : n(r, y) = k\} \quad \text{and} \quad B_k := \{y \in S(t) : n(r, y) \geq k\}.$$

Let $B'_k \subset B_k$ be compact with $H^{n-1}(B'_k) \geq H^{n-1}(B_k)/2$ and consider the path family $\Gamma_k := \Delta(F, B'_k; B(t))$. Each $\gamma \in \Gamma_k$ has $k$ liftings $\alpha_1, \ldots, \alpha_k$ under $f$ which connect a point in $\overline{B}(r)$ to $S(\theta r)$ and have the properties stated in Lemma 4.3; cf. [24, Section II.3]. Let $\Gamma_k^*$ be the family of all these liftings. Then

$$M(\Gamma_k) \leq \frac{K_I(f)}{k} M(\Gamma_k^*)$$

by Lemma 4.3. By Lemma 4.2 and (4.1) we have

$$M(\Gamma_k^*) \leq M\big(\Delta\big(S(r), S(\theta r); B(\theta r)\backslash\overline{B}(r)\big)\big) = \omega_{n-1}(\log\theta)^{1-n}.$$

Combining the last two inequalities we obtain

$$M(\Gamma_k) \leq \frac{K_I(f)}{k}\omega_{n-1}(\log\theta)^{1-n}. \tag{5.1}$$

Applying Lemma 4.5 with $F_1 = F$, $F_2 = B'_k$, $u = t/2$ and $v = t$ and noting that $M(\Delta(F, S(t); B(t))) = \operatorname{cap}(B(t), F)$ by Lemma 4.1 we obtain

$$M(\Gamma_k) \geq 3^{-n} \min\left\{\operatorname{cap}(B(t), F), M(\Delta(B'_k, S(t/2); B(t))), c_n \log 2\right\}. \tag{5.2}$$

Let $k_0$ be the integer part of

$$\frac{3^n K_I(f)\omega_{n-1}(\log\theta)^{1-n}}{\min\left\{\operatorname{cap}(B(t), F), c_n \log 2\right\}}.$$

Using (5.1) we see that for $k > k_0$ the minimum in the right side of (5.2) is attained by the term in the middle so that

$$M(\Gamma_k) \geq 3^{-n} M(\Delta(B'_k, S(t/2); B(t)))$$

and thus

$$M(\Gamma_k) \geq \alpha \left(\log\left(\frac{\beta t^{n-1}}{H^{n-1}(B'_k)}\right)\right)^{1-n}$$

by Lemma 4.6. Together with (5.1) we obtain

$$\alpha \left(\log\left(\frac{\beta t^{n-1}}{H^{n-1}(B'_k)}\right)\right)^{1-n} \leq \frac{K_I(f)}{k}\omega_{n-1}(\log\theta)^{1-n} \quad \text{for } k > k_0.$$

This yields

$$\frac{H^{n-1}(B'_k)}{t^{n-1}} \leq \beta \exp\left(-\tau \log\theta \left(\frac{k}{K_I(f)}\right)^{1/(n-1)}\right) \quad \text{for } k > k_0, \tag{5.3}$$

where $\tau = (\omega_{n-1}/\alpha)^{1/(1-n)}$ depends only on $n$. Now

$$\begin{aligned}
\nu(r, S(z,t)) =& \frac{1}{\omega_{n-1}t^{n-1}} \sum_{k=1}^{\infty} k H^{n-1}(A_k) \\
=& \frac{1}{\omega_{n-1}t^{n-1}} \sum_{k=1}^{\infty} k \left(H^{n-1}(B_k) - H^{n-1}(B_{k+1})\right) \\
=& \frac{1}{\omega_{n-1}t^{n-1}} \sum_{k=1}^{\infty} H^{n-1}(B_k) \\
\leq& \frac{1}{\omega_{n-1}t^{n-1}} \left(k_0 H^{n-1}(S(t)) + 2\sum_{k=k_0+1}^{\infty} H^{n-1}(B'_k)\right) \\
=& k_0 + \frac{2}{\omega_{n-1}t^{n-1}} \sum_{k=k_0+1}^{\infty} H^{n-1}(B'_k).
\end{aligned} \tag{5.4}$$

By (5.3) we have

$$
\begin{aligned}
&\frac{2}{\omega_{n-1}t^{n-1}}\sum_{k=k_0+1}^{\infty} H^{n-1}(B'_k) \\
&\leq \frac{2\beta}{\omega_{n-1}}\sum_{k=1}^{\infty}\exp\left(-\tau\log\theta\left(\frac{k}{K_I(f)}\right)^{1/(n-1)}\right) \\
&\leq \frac{2\beta}{\omega_{n-1}}\int_{u=0}^{\infty}\exp\left(-\tau\log\theta\left(\frac{u}{K_I(f)}\right)^{1/(n-1)}\right)du \\
&= C_1\, K_I(f)(\log\theta)^{1-n},
\end{aligned}
\tag{5.5}
$$

with

$$C_1 := \frac{2\beta}{\omega_{n-1}}\int_{u=0}^{\infty}\exp\left(-\tau u^{1/(n-1)}\right)du$$

depending only on $n$. Noting that

$$\operatorname{cap}(B(t),F)\leq \operatorname{cap}\big(B(t),\overline{B}(t/2)\big)=\omega_{n-1}(\log 2)^{1-n} \tag{5.6}$$

by (4.1) we see that

$$k_0\leq C_2\frac{K_I(f)(\log\theta)^{1-n}}{\operatorname{cap}(B(t),F)} \tag{5.7}$$

for some constant $C_2$ depending only on $n$. Combining (5.4), (5.5) and (5.7) we obtain

$$\nu(r,S(z,t))\leq\left(C_1+\frac{C_2}{\operatorname{cap}(B(t),F)}\right)K_I(f)(\log\theta)^{1-n},$$

which together with (5.6) yields (3.1) with $C := \omega_{n-1}(\log 2)^{1-n}C_1 + C_2$. $\square$

## 6. Proof of Theorems 1.1–1.3

*Proof of Theorem 1.1.* First we note that

$$A\big(\overline{\mathbb{R}^n},f^k\big)=\deg(f^k)=(\deg(f))^k$$

for $k\in\mathbb{N}$. Next we observe that there exists a constant $L$ depending only on $n$ such $\overline{\mathbb{R}^n}$ can be covered by $Lk^n$ balls of chordal radius $1/k$, for all $k\in\mathbb{N}$. Hence for each $k\in\mathbb{N}$ there exists $x_k\in\overline{\mathbb{R}^n}$ such that

$$A\big(\overline{B}_\chi(x_k,1/k)\,,f^k\big)\geq\frac{1}{Lk^n}A\big(\overline{\mathbb{R}^n},f^k\big)=\frac{(\deg(f))^k}{Lk^n}.$$

The sequence $(x_k)$ has a convergent subsequence, say $x_{k_j}\to x$. We will show that (1.1) holds for every neighborhood $U$ of $x$.

Suppose that this is not the case. Then there exists a set $F$ of positive capacity and $\delta>0$ such that $O^+\big(\overline{B}_\chi(x,2\delta)\big)\subset\overline{\mathbb{R}^n}\setminus F$. Since $K_I(f^k)\leq K_I(f)^k$ by (2.3) we deduce from Theorem 3.2 and the remark following it that

$$A\big(\overline{B}_\chi(x,\delta),f^k\big)\leq CK_I(f)^k$$

for some constant $C$. On the other hand, for sufficiently large $k$ we have $\overline{B}_\chi(x_{k_j}, 1/k_j) \subset \overline{B}_\chi(x, \delta)$ and thus

$$A\big(\overline{B}_\chi(x,\delta), f^{k_j}\big) \geq A\big(\overline{B}_\chi\left(x_{k_j}, 1/k_j\right), f^{k_j}\big) \geq \frac{(\deg(f))^{k_j}}{Lk_j^n}.$$

The last two inequalities yield

$$\frac{(\deg(f))^{k_j}}{Lk_j^n} \leq CK_I(f)^{k_j}.$$

For large $j$ this contradicts the assumption that $\deg(f) > K_I(f)$. □

*Proof of Theorem 1.2.* Denote by $J_1(f)$ the set where the iterates are not normal and by $J_2(f)$ the set given by Definition 1.1.

If $x \in J_1(f)$ and $U$ is a neighborhood of $x$, then, as already mentioned in the introduction, $\overline{\mathbb{R}^n} \setminus O^+(U)$ is finite by Miniowitz's theorem and thus (1.1) holds. Hence $x \in J_2(f)$.

If $x \in F(f) := \overline{\mathbb{R}^n} \setminus J_1(f)$, then there exists a neighborhood $U$ of $x$ satisfying $U \subset F(f)$. By the complete invariance of $F(f)$ and $J_1(f)$ we have $O^+(U) \subset F(f)$ and thus $\overline{\mathbb{R}^n} \setminus O^+(U) \supset J_1(f)$. By the result of Fletcher and Nicks [9] already mentioned in the introduction, we have $\dim J_1(f) > 0$ and thus $\dim\big(\overline{\mathbb{R}^n} \setminus O^+(U)\big) > 0$. Hence $\operatorname{cap}\big(\overline{\mathbb{R}^n} \setminus O^+(U)\big) > 0$ by Lemma 4.7. We conclude that $x \notin J_2(f)$.

Altogether we see that $J_1(f) = J_2(f)$. □

*Proof of Theorem 1.3.* It is easy to see that $J(f) \cap A(\xi) = \emptyset$. Let $x \in J(f)$. Then $x \notin A(\xi)$. Suppose now that $x \notin \partial A(\xi)$. Then there exists a neighborhood $U$ of $x$ such that $U \cap A(\xi) = \emptyset$ and thus $O^+(U) \cap A(\xi) = \emptyset$. In particular, $\overline{\mathbb{R}^n} \setminus O^+(U) \supset \overline{B}_\chi(\xi, \varepsilon)$ for some $\varepsilon > 0$. Since $\overline{B}_\chi(\xi, \varepsilon)$ has positive capacity, this is a contradiction. □

## 7. Local distortion of quasiregular maps

The following result can be found in [24, Theorem III.4.7].

**Lemma 7.1.** *Let $\Omega \subset \mathbb{R}^n$ be open, $f : \Omega \to \mathbb{R}^n$ be quasiregular and $x \in \Omega$. Then there exists $A, B, r > 0$ such that*

$$A|x-y|^\nu \leq |f(x) - f(y)| \leq B|x-y|^\mu \quad \text{for } y \in B(x,r), \tag{7.1}$$

*where $\nu := (K_O(f) i(x,f))^{1/(n-1)}$ and $\mu := (i(x,f)/K_I(f))^{1/(n-1)}$.*

The right inequality of (7.1) is due to Martio [15]. It was one of the main tools used by Fletcher and Nicks [8] in their study of quasiregular maps of polynomial type. The left inequality of (7.1) is due to Srebro [27]. It will not be needed in the sequel and is listed here only for completeness.

Lemma 7.1 extends to the case where the domain and range of $f$ are in $\overline{\mathbb{R}^n}$, provided the Euclidean metric is replaced by the chordal metric. Clearly the number $r$ in Lemma 7.1 depends on $x$, as the left inequality of (7.1) implies that $f(x) \neq f(y)$ for $y \in B(x,r) \setminus \{x\}$. In the proof, the dependence on $r$ comes in because $f$ is considered in a normal neighborhood of $x$. This is, by definition, a neighborhood $U$ compactly contained in $\Omega$ such that $f(\partial U) = \partial f(U)$ and $U \cap f^{-1}(f(x)) = \{x\}$.

For a quasiregular map $f : \overline{\mathbb{R}^n} \to \overline{\mathbb{R}^n}$ we put $B_f^* := \{x \in \overline{\mathbb{R}^n} : i(x, f) = \deg(f)\}$. For $x \in B_f^*$ we have $f^{-1}(f(x)) = \{x\}$ and thus the last condition in the definition of a normal neighborhood is automatically satisfied. Using this observation and noting that $B_f^*$ is compact we can deduce from the proof of Lemma 7.1 in [24] that we may take the same values $A$, $B$ and $r$ for all $x \in B_f^*$. Thus we obtain the following result.

**Lemma 7.2.** *Let $f : \overline{\mathbb{R}^n} \to \overline{\mathbb{R}^n}$ be quasiregular. Then there exists $A, B, r > 0$ such that if $x \in B_f^*$, then*

$$A\,\chi(x, y)^\nu \le \chi(f(x), f(y)) \le B\,\chi(x, y)^\mu \quad \text{for } y \in B_\chi(x, r), \tag{7.2}$$

*where $\nu := (K_O(f)\deg(f))^{1/(n-1)}$ and $\mu := (\deg(f)/K_I(f))^{1/(n-1)}$.*

We shall only need the right inequality of (7.2). This inequality can also be deduced from the following result.

**Lemma 7.3.** *Let $f : \Omega \to \mathbb{R}^n$ be quasiregular and put*

$$\mu(m) := \left(\frac{m}{K_I(f)}\right)^{1/(n-1)}$$

*for $m \in \mathbb{N}$. Let $X \subset \Omega$ compact. Then there exists $C, r > 0$ such that*

$$|f(x) - f(y)| \le C|x - y|^{\mu(i(x,f))} \quad \text{for } x \in X \text{ and } y \in B(x, r).$$

Lemma 7.3 is probably known, but I have not been able to find it in the literature. Therefore we will include a proof of Lemma 7.3 at the end of this section. In fact, we will see that Lemma 7.3 is a simple corollary of the following Proposition 7.1 which may be of independent interest. Proposition 7.1 will also be used in section 11.

In order to state this proposition, we first introduce some terminology. Let $f : \Omega \to \mathbb{R}^n$ be quasiregular and let $U$ be a domain compactly contained in $\Omega$. Then $U$ is called a *normal domain* for $f$ if $f(\partial U) = \partial f(U)$. The normal neighborhood of a point $x$ mentioned above is thus a normal domain $U$ satisfying $U \cap f^{-1}(f(x)) = \{x\}$. For $x \in \Omega$ and $s > 0$ we denote by $U(x, f, s)$ the component of $f^{-1}(B(f(x), s))$ that contains $x$ and by $\overline{U}(x, f, s)$ its closure. We note that if $U(x, f, s)$ is compactly contained in $\Omega$, then $U(x, f, s)$ is a normal domain and thus $f$ is a proper map from $U(x, f, s)$ onto $B(f(x), s)$. We denote the degree of this map by $d(x, f, s)$.

**Proposition 7.1.** *For $M, n \in \mathbb{N}$ and $K \ge 1$ there exists $c, \eta > 0$ with the following properties: let $f : \Omega \to \mathbb{R}^n$ be $K$-quasiregular, $x \in \Omega$, $\sigma > 0$ and $0 < s \le \eta\sigma$. Suppose that $U(x, f, \sigma)$ is compactly contained in $\Omega$ and $d(x, f, \sigma) \le M$. Then there exists an integer $m$, depending on $x$ and $s$ and satisfying $d(x, f, s) \le m \le M$, such that*

$$U(x, f, s/\eta) \supset \overline{B}\big(x, c\, s^{1/\mu(m)}\big) \tag{7.3}$$

*and*

$$n\left(\overline{B}\left(x, c\, s^{1/\mu(m)}\right), y\right) \le m \quad \text{for } y \in B(f(x), s), \tag{7.4}$$

*with $\mu(m)$ defined as in Lemma 7.3.*

The proof of Proposition 7.1 requires the following result known as the $K_O$-inequality [24, Theorem II.10.9]. Here a condenser $(G, C)$ is called *normal* for a quasiregular map $f$ if $G$ is a normal domain for $f$.

**Lemma 7.4.** *Let $(G, C)$ be a normal condenser for a quasiregular map $f$. If $N \in \mathbb{N}$ is such that $\operatorname{card}(f^{-1}(y) \cap G) \le N$ for all $y \in f(G)$, then*

$$\operatorname{cap}(G, C) \le N\, K_O(f) \operatorname{cap}(f(G), f(C)).$$

We shall also need the following lemma which can be deduced from [33, Lemma 5.42]. Here a condenser $(G, C)$ is called *ringlike* if $C$ and $\mathbb{R}^n \setminus G$ are connected.

**Lemma 7.5.** *There exists $\kappa > 0$ depending only on the dimension such that if $(G, C)$ is a ringlike condenser with $\operatorname{cap}(G, C) < \kappa$, then $\overline{B}(x, \operatorname{diam} C) \subset G$ for all $x \in C$.*

*Proof of Proposition 7.1.* With $\rho := \sup\{r > 0 : B(x, r) \subset U(x, f, \sigma)\}$ we have $f(B(x, \rho)) \subset B(f(x), \sigma)$. Put $\eta := 1/L^{M+1}$ with a constant $L > 1$ to be determined later. Clearly, the function $t \mapsto d(f, x, t)$ is non-decreasing and takes values in $\{1, 2, \dots, M\}$. It follows that for $0 < s \le \eta\sigma = \sigma/L^{M+1}$ there exists $t(s) \in [s, L^M s]$ such that $t \mapsto d(x, f, t)$ is constant in $[t(s), Lt(s)]$. We put $m := d(x, f, t(s))$. From Lemma 7.4 we deduce that

$$\begin{aligned} \operatorname{cap}\left(U(x, f, Lt(s)), \overline{U}(x, f, t(s))\right) &\le m K_O(f) \operatorname{cap}\left(B(x, Lt(s)), \overline{B}(x, t(s))\right) \\ &= m K_O(f) \omega_{n-1} (\log L)^{1-n}. \end{aligned}$$

Since $m \le M$ we see that the right hand side is less than the constant $\kappa$ from Lemma 7.5, if $L$ is chosen large enough. Denoting by $\tau(s)$ the diameter of $U(x, f, t(s))$ we conclude that

$$\overline{B}(x, \tau(s)) \subset U(x, f, Lt(s)) \subset U(x, f, L^{M+1} s) = U(x, f, s/\eta). \tag{7.5}$$

In particular, $\overline{B}(x, \tau(s)) \subset U(x, f, \sigma)$ and thus $\tau(s) < \rho$.

By Lemma 4.4 we have

$$\operatorname{cap}\left(B(f(x), \sigma), \overline{B}(f(x), t(s))\right) \le \frac{K_I(f)}{m} \operatorname{cap}\left(U(x, f, \sigma), \overline{U}(x, f, t(s))\right).$$

Now $B(x, \rho) \subset U(x, f, \sigma)$ and $\overline{U}(x, f, t(s)) \subset \overline{B}(x, \tau(s))$. Noting that $\tau(s) < \rho$ we obtain

$$\operatorname{cap}\left(B(f(x), \sigma), \overline{B}(f(x), t(s))\right) \le \frac{K_I(f)}{m} \operatorname{cap}\left(B(x, \rho), \overline{B}(x, \tau(s))\right).$$

Using (4.1) we deduce that

$$\left(\log \frac{\rho}{\tau(s)}\right)^{n-1} \le \frac{K_I(f)}{m} \left(\log \frac{\sigma}{t(s)}\right)^{n-1}.$$

Solving this inequality for $\tau(s)$ we obtain

$$\tau(s) \geq \rho\left(\frac{t(s)}{\sigma}\right)^{1/\mu(m)} \geq c\, s^{1/\mu(m)}$$

for some positive constant $c$. Together with (7.5) this yields (7.3).

Since $\overline{B}(x,\tau(s)) \subset U(x,f,Lt(s))$ by (7.5) we have $n\big(\overline{B}(x,\tau(s)),y\big) \leq m$ for $y \in B(f(x),Lt(s))$ and thus, in particular, for $y \in B(f(x),s)$. This is (7.4). □

*Proof of Lemma 7.3.* Since $X$ is compact, there exist $M \in \mathbb{N}$ and $\sigma > 0$ such that $U(x,f,\sigma)$ is compactly contained in $\Omega$ and $d(x,f,\sigma) \leq M$ for all $x \in X$. Let $0 < s \leq \eta\sigma$ and choose $m$ according to Lemma 7.3. Since $m \geq d(x,f,s) \geq i(x,f)$ we have $s^{1/\mu(m)} \geq s^{1/\mu(i(x,f))}$ and hence we deduce from (7.3) that

$$U(x,f,s/\eta) \supset B\big(x, c\, s^{1/\mu(i(x,f))}\big)\,.$$

Thus $|f(y)-f(x)| \leq s/\eta$ if $|y-x| = c\, s^{1/\mu(i(x,f))}$. Solving the last equation for $s$ and substituting the result into the estimate for $|f(y)-f(x)|$ we obtain the conclusion. □

## 8. Proof of Theorem 1.4

Let $x \in E(f)$. Since $\operatorname{card} f^{-1}(A) \geq \operatorname{card} A$ for every finite subset $A$ of $\overline{\mathbb{R}^n}$ and $f^{-1}(O^-(x) \cup \{x\}) = O^-(x)$ we have

$$\operatorname{card} O^-(x) = \operatorname{card}(f^{-1}(O^-(x) \cup \{x\})) \geq \operatorname{card}(O^-(x) \cup \{x\})$$

and thus $x \in O^-(x)$. Hence $x$ is periodic.

Moreover, the argument shows that $\operatorname{card} f^{-1}(y) = 1$ for all $y \in E(f)$. Choosing $p \in \mathbb{N}$ such that $f^p(x) = x$ we thus have $f^{-p}(x) = \{x\}$. This implies that $i(x,f^p) = \deg(f^p)$. With $B$ and $r$ as in Lemma 7.2, applied to $f^p$, we obtain

$$\chi(f^p(y),x) = \chi(f^p(y),f^p(x)) \leq B\,\chi(y,x)^\mu$$

for $\chi(y,x) < r$, where

$$\mu := \left(\frac{\deg(f^p)}{K_I(f^p)}\right)^{1/(n-1)}.$$

Since $\deg(f^p) = \deg(f)^p > K_I(f)^p \geq K_I(f^p)$ by (2.3) we have $\mu > 1$. Hence there exists $\delta > 0$ depending only on $B$, $r$ and $\mu$ such that

$$\chi(f^p(y),x) \leq \frac{1}{2}\chi(y,x)$$

for $\chi(y,x) \leq \delta$. Thus $x$ is an attracting periodic point. Moreover, we have $B_\chi(x,\delta) \subset A(x)$, which implies that the chordal distance between two points in $E(f)$ is at least $\delta$. Thus $E(f)$ is finite. □

## 9. Hausdorff measure of invariant sets

Let $\eta > 0$. An increasing, continuous function $h : (0, \eta] \to (0, \infty)$ satisfying $\lim_{t \to 0} h(t) = 0$ is called a *gauge function*. For a set $X \subset \mathbb{R}^n$ and a gauge function $h$ the *Hausdorff measure* $H_h(X)$ is defined by

$$H_h(X) := \lim_{\delta \to 0} \inf_{(X_j)} \sum_{j=1}^{\infty} h(\operatorname{diam} X_j),$$

where the infimum is taken over all sequences $(X_j)$ of subsets of $\mathbb{R}^n$ such that $X \subset \bigcup_{j=1}^{\infty} X_j$ and $\operatorname{diam} X_j < \delta$ for all $j$. The $d$-dimensional Hausdorff measure $H^d(X)$ considered already corresponds to the function $h(t) = t^d$, up to a normalization factor.

Recall that for $X \subset \mathbb{R}^n$ and $f : X \to \mathbb{R}^n$ an increasing, continuous function $\omega : [0, \infty) \to [0, \infty)$ is called a *modulus of continuity* for $f$ if $\omega(0) = 0$ and $|f(x) - f(y)| \leq \omega(|x - y|)$ for all $x, y \in X$. If this holds with $\omega(t) = Lt^{\alpha}$ where $L, \alpha > 0$, then $f$ is said to be *Hölder continuous* with exponent $\alpha$ and in the special case that $\alpha = 1$ we say that $f$ is *Lipschitz continuous* with *Lipschitz constant* $L$. Identifying $\overline{\mathbb{R}^n}$ with $S^n(1) \subset \mathbb{R}^{n+1}$ we also use this terminology for $X \subset \overline{\mathbb{R}^n}$ and $f : X \to \overline{\mathbb{R}^n}$. (Equivalently, we can replace the Euclidean metric by the chordal metric in the definition of the modulus of continuity.)

**Theorem 9.1.** *Let $X \subset \mathbb{R}^n$ be compact and let $f : X \to X$ be a continuous function with modulus of continuity $\omega$.*

*Suppose that there exists $m \in \mathbb{N}$, $m \geq 2$, and $\delta > 0$ such that each $y \in X$ has $m$ preimages $x_1, \dots, x_m$ satisfying $|x_i - x_j| \geq \delta$ for $i \neq j$. If $h$ is a gauge function such that $\omega^k(h^{-1}(1/m^k)) \leq \delta/2$ for all large $k$, then $H_h(X) > 0$.*

We give two corollaries dealing with Lipschitz and Hölder continuous maps.

**Corollary 9.1.** *Let $X \subset \mathbb{R}^n$ be compact and let $f : X \to X$ be continuous such that each $y \in X$ has $m$ preimages $x_1, \dots, x_m$ satisfying $|x_i - x_j| \geq \delta$ for $i \neq j$. If $f$ satisfies a Lipschitz condition with Lipschitz constant $L > 1$, then*

$$\dim X \geq \frac{\log m}{\log L}.$$

**Corollary 9.2.** *Let $X \subset \mathbb{R}^n$ be compact and let $f : X \to X$ be continuous such that each $y \in X$ has $m$ preimages $x_1, \dots, x_m$ satisfying $|x_i - x_j| \geq \delta$ for $i \neq j$. If $f$ satisfies a Hölder condition with exponent $\alpha < 1$, then $H_h(X) > 0$ for*

$$h(t) = \left( \log \frac{1}{t} \right)^{(\log m)/(\log \alpha)}.$$

For the proof of Theorem 9.1 we need the following version of the so-called mass distribution principle; see [20, Theorem 7.6.1].

**Lemma 9.1.** *Let $X \subset \mathbb{R}^n$ be compact and let $h$ be a gauge function. Suppose that there exist a probability measure $\mu$ supported on $X$ and $c, \eta > 0$ such that $\mu(B(x, r)) \leq c\, h(r)$ for $0 < r \leq \eta$ and all $x \in X$. Then $H_h(X) > 0$.*

*Proof of Theorem 9.1.* For each finite subset $E_0$ of $X$ we can choose a finite subset $E_1$ of $f^{-1}(E_0)$ such that each point in $E_0$ has $m$ preimages in $E_1$, with $|x - x'| \geq \delta$ if $x, x' \in E_1$ with $f(x) = f(x')$. Clearly, $\operatorname{card} E_1 = m \operatorname{card} E_0$. Beginning with $E_0 = \{y\}$ for some fixed $y \in X$ and performing this process repeatedly we obtain a sequence $(E_k)$ of sets with $\operatorname{card} E_k = m^k$ such that each point in $E_{k-1}$ has $m$ preimages in $E_k$, with $|x - x'| \geq \delta$ if $x, x' \in E_k$ with $f(x) = f(x')$. We denote by $\delta_x$ the Dirac measure at a point $x$ and, for $k \geq 0$, define the measure $\mu_k$ by

$$\mu_k := \frac{1}{m^k} \sum_{x \in E_k} \delta_x.$$

The sequence $(\mu_k)$ has a subsequence which converges with respect to the weak$^*$-topology, say $\mu_{k_j} \to \mu$; see, e.g., [35, Theorem 6.5].

For $x \in X$ and $0 < r \leq \delta/2$ we have

$$\mu_{k+1}(B(x, r)) \leq \frac{1}{m} \mu_k(B(f(x), \omega(r))).$$

Thus

$$\mu_{k+l}(B(x, r)) \leq \frac{1}{m^l} \mu_k(B(f^l(x), \omega^l(r)))$$

for $l \in \mathbb{N}$ as long as $\omega^{l-1}(r) \leq \delta/2$.

If $\omega^j(r) \leq \delta/2$ for all $j \in \mathbb{N}$, then $\mu(B(x, r)) = 0$ for all $x \in X$ and thus $\mu \equiv 0$, which is a contradiction. Thus there exists $j \in \mathbb{N}$ depending on $r$ such that $\omega^{j-1}(r) \leq \delta/2 < \omega^j(r)$. Denoting by $\tau$ the inverse function of $\omega$ we thus have $\tau^j(\delta/2) < r \leq \tau^{j-1}(\delta/2)$. Note that $j$ is large when $r$ is small. By hypothesis we thus have $\omega^j(h^{-1}(1/m^j)) \leq \delta/2$ and hence $h(\tau^j(\delta/2)) \geq 1/m^j$ for small $r$, say for $0 < r \leq \eta$. For $k \geq j$ we thus obtain

$$\mu_k(B(x, r)) \leq \frac{1}{m^j} \mu_{k-j}(B(f^j(x), \omega^j(r))) \leq \frac{1}{m^j} \leq h(\tau^j(\delta/2)) \leq h(r).$$

We conclude that $\mu(B(x, r)) \leq h(r)$ for $x \in X$ and $0 < r \leq \eta$ so that the conclusion follows from Lemma 9.1. $\square$

*Proof of Corollaries 9.1 and 9.2.* Let $\omega(t) = L\, t^\alpha$. By induction we find that

$$\omega^k(t) = L^{p_k} t^{\alpha^k} \quad \text{with } p_k = \sum_{j=0}^{k-1} \alpha^j$$

for $k \in \mathbb{N}$.

First we consider the case that $\alpha = 1$. Then $p_k = k$ so that $\omega^k(t) = L^k t$. Define $h_1(t) := (2t/\delta)^{(\log m)/(\log L)}$. Then

$$h_1^{-1}\left(\frac{1}{m^k}\right) = \frac{\delta}{2} \left(\frac{1}{m^k}\right)^{(\log L)/(\log m)} = \frac{\delta}{2L^k}.$$

Hence $\omega^k(h_1^{-1}(1/m^k)) = \delta/2$. Thus $H_{h_1}(X) > 0$ by Theorem 9.1, and Corollary 9.1 follows.

Now we consider the case that $\alpha < 1$. Then $p_k = (1-\alpha^k)/(1-\alpha) \sim 1/(1-\alpha)$ as $k \to \infty$ so that $\omega^k(t) \le c\, t^{\alpha^k}$ for some $c > 0$. With $b := \log(2c/\delta)$ we now define

$$h_2(t) := \left(\frac{1}{b}\log\frac{1}{t}\right)^{(\log m)/(\log \alpha)}.$$

Then $h_2^{-1}(t) = \exp\bigl(-b\, t^{(\log\alpha)/(\log m)}\bigr)$ and thus

$$h_2^{-1}\left(\frac{1}{m^k}\right) = \exp\left(-b\left(\frac{1}{m^k}\right)^{(\log\alpha)/(\log m)}\right) = \exp\bigl(-b\alpha^{-k}\bigr).$$

Hence $\omega^k(h_2^{-1}(1/m^k)) \le c\, e^{-b} = \delta/2$. Thus $H_{h_2}(X) > 0$ by Theorem 9.1, and Corollary 9.2 follows. □

## 10. Proof of Theorems 1.5–1.8

*Proof of Theorem 1.5.* Suppose that there exists an open set $U$ intersecting $J(f)$ such that $O^+(U) \not\supset \overline{\mathbb{R}^n} \setminus E(f)$. Then there exists $z \in \overline{\mathbb{R}^n} \setminus (O^+(U) \cup E(f))$. We note that $O^-(z)$ is infinite since $z \notin E(f)$. Let $X$ be the set of limit points of $O^-(z)$. Then $X$ is a non-empty, closed and completely invariant subset of $\overline{\mathbb{R}^n} \setminus O^+(U)$. Moreover, $X \cap E(f) = \emptyset$ by Theorem 1.4.

First we show that for each $x \in X$ there exist $m \in \mathbb{N}$ such that $f^{-m}(x)$ contains at least two points. Otherwise there exist $y_0 \in X$ such that for all $k \in \mathbb{N}$ we have $f^{-k}(y_0) = \{y_k\}$ for some $y_k \in X$. With $B_f^*$ as in Lemma 7.2 we find that $y_k \in B_f^*$ for all $k \in \mathbb{N}$. It follows from Lemma 7.2 that there exists $\delta > 0$ such that if $x \in B_f^*$ and $y \in \overline{\mathbb{R}^n}$ with $\chi(x,y) < \delta$, then $\chi(f(x), f(y)) \le \chi(x,y)/2 < \delta/2$. Now there exists $N \in \mathbb{N}$ such that if $x_1, \dots, x_N \in \overline{\mathbb{R}^n}$, then there exist $k, l \in \{1, \dots, N\}$ with $k \ne l$ such that $\chi(x_k, x_l) < \delta$. In particular, for each $m \in \mathbb{N}$ there exist $k, l \in \{1, \dots, N\}$ with $k \ne l$ such that $\chi(y_{m+k}, y_{m+l}) < \delta$. It follows that

$$\chi(y_k, y_l) = \chi(f^m(y_{m+k}), f^m(y_{m+l})) \le \frac{1}{2^m}\chi(y_{m+k}, y_{m+l}) < \frac{1}{2^m}\delta. \tag{10.1}$$

On the other hand, since $y_0 \notin E(f)$, all the points $y_k$ are distinct and thus

$$\eta := \min_{1 \le k < l \le N} \chi(y_k, y_l) > 0. \tag{10.2}$$

Choosing $m$ such that $2^m > \delta/\eta$ we obtain a contradiction from (10.1) and (10.2). Thus for each $x \in X$ there exist $m \in \mathbb{N}$ such that $f^{-m}(x)$ contains at least two points.

Noting that $f$ is an open map we deduce that for every $x \in X$ there exist $m(x) \in \mathbb{N}$, $\delta(x) > 0$ and a neighborhood $U(x)$ of $x$ such that if $y \in U(x)$, then $f^{-m(x)}(y)$ contains two points whose chordal distance is at least $\delta(x)$. The compact set $X$ can be covered by finitely many such neighborhoods, say $X \subset \bigcup_{j=1}^k U(x_j)$. Let $m := \max_j m(x_j)$. Since $f$ and its iterates are continuous there exists $\delta > 0$ such that $\chi(f^{m-m(x_j)}(x), f^{m-m(x_j)}(y)) < \delta(x_j)$ for $x, y \in X$ with $\chi(x,y) < \delta$ and $j \in \{1, \dots, k\}$. We find that for each $x \in X$ the preimage $f^{-m}(x)$ contains two points whose chordal distance is at least $\delta$.

It now follows from Corollary 9.1, which we may apply by considering the subset $X$ of $\overline{\mathbb{R}^n}$ as a subset of $\mathbb{R}^{n+1}$, that $\dim X > 0$. Thus $\operatorname{cap} X > 0$ by Lemma 4.7. On the other hand, since $X \subset \overline{\mathbb{R}^n} \setminus O^+(U)$, we have $\operatorname{cap} X = 0$ by Theorem 1.1. This is a contradiction.

Hence $O^+(U) \supset \overline{\mathbb{R}^n} \setminus E(f)$ if $U$ is an open set intersecting $J(f)$. Since $J(f)$ is completely invariant and $J(f) \cap E(f) = \emptyset$ by Theorem 1.4, we also deduce that $O^+(U \cap J(f)) = J(f)$. □

*Proof of Theorem 1.6.* Let $x \in \overline{\mathbb{R}^n} \setminus E(f)$ and suppose that $J(f) \not\subset \overline{O^-(x)}$ so that there exists $y \in J(f) \setminus \overline{O^-(x)}$. Then $y$ has a neighborhood $U$ satisfying $U \cap \overline{O^-(x)} = \emptyset$. It follows that $O^+(U) \cap \overline{O^-(x)} = \emptyset$. Thus $O^-(x) \subset E(f)$ by Theorem 1.5. Hence $x \in E(f)$, which is a contradiction. We deduce that $J(f) \subset \overline{O^-(x)}$ for $x \in \overline{\mathbb{R}^n} \setminus E(f)$.

Since $J(f) \cap E(f) = \emptyset$ by Theorem 1.4 we deduce that $J(f) \subset \overline{O^-(x)}$ holds in particular for $x \in J(f)$. On the other hand, if $x \in J(f)$, then $O^-(x) \subset J(f)$ since $J(f)$ is completely invariant and hence $\overline{O^-(x)} \subset J(f)$ since $J(f)$ is closed. It follows that $\overline{O^-(x)} = J(f)$ for $x \in J(f)$. □

*Proof of Theorem 1.7.* We use the same method as in the proof of Theorem 1.5. In fact, here the argument is even a little simpler.

Noting again that $J(f) \cap E(f) = \emptyset$ by Theorem 1.4 we see as in the proof of Theorem 1.5 that there exist $m \in \mathbb{N}$ and $\delta > 0$ such that for each $x \in J(f)$ the preimage $f^{-m}(x)$ contains two points whose chordal distance is at least $\delta$. Corollary 9.1 now yields the conclusion. □

The proof of Theorem 1.8 requires the following strengthening of Lemma 4.7; see [34].

**Lemma 10.1.** *Let $X \subset \mathbb{R}^n$ be compact and $\varepsilon > 0$. If $H_h(X) > 0$ for*

$$h(t) = \left(\log \frac{1}{t}\right)^{1-n-\varepsilon}, \tag{10.3}$$

*then* $\operatorname{cap} X > 0$.

*Proof of Theorem 1.8.* The argument is similar to that used in the proof of Theorem 1.7 (and Theorem 1.5). However, since we assume that $J(f) \cap B_f = \emptyset$, we now find that there exists $\delta > 0$ such that each $x \in J(f)$ has $d := \deg(f)$ preimages, any two of which have chordal distance at least $\delta$. Moreover, $f$ is Hölder continuous with exponent $\alpha := K_I(f)^{1/(1-n)}$; see [24, Theorem III.1.11] or Lemma 7.3. Corollary 9.2 now yields that $H_h(J(f)) > 0$ for

$$h(t) = \left(\log \frac{1}{t}\right)^{(\log d)/(\log \alpha)} = \left(\log \frac{1}{t}\right)^{((1-n)\log d)/(\log K_I(f))}.$$

Since we assume that $d > K_I(f)$ we have

$$\frac{(1-n)\log d}{\log K_I(f)} = 1 - n - \varepsilon$$

for some $\varepsilon > 0$. Now the conclusion follows from Lemma 10.1. □

## 11. A conjecture about the capacity of invariant sets

We conjectured in the introduction that the hypothesis that $f$ is Lipschitz continuous can be omitted in Theorem 1.5 and 1.6. The proof of these theorems shows that this conjecture would follow from the next one.

**Conjecture 11.1.** *Let $f : \overline{\mathbb{R}^n} \to \overline{\mathbb{R}^n}$ be quasiregular with $\deg(f) > K_I(f)$. Let $X \subset \overline{\mathbb{R}^n} \setminus E(f)$ be compact and completely invariant. Then* $\operatorname{cap} X > 0$.

While we have been unable to prove this conjecture, we give some arguments in favor of it. The idea is to consider, for fixed $y \in X$, the measures

$$\lambda_k := \frac{1}{\deg(f)^k} \sum_{x \in f^{-k}(y)} i\big(x, f^k\big)\, \delta_x. \tag{11.1}$$

These measures play an important role in complex dynamics; cf. [28, Section 161]. Let $\lambda$ be the limit of a convergent subsequence. Assuming without loss of generality that $X \subset \mathbb{R}^n$, we deduce from Lemmas 9.1 and 10.1 that it suffices to prove that $\lambda(B(x, r)) \le h(r)$ for $0 < r \le \eta$ and all $x \in X$, where $h$ is given by (10.3).

Now it follows from Proposition 7.1 that if $s$ is sufficiently small and $x \in X$, then there exists $m$ satisfying $i(x, f) \le m \le \deg(f)$ such that

$$\lambda\big(B\big(x, cs^{1/\mu(m)}\big)\big) \le \frac{m}{\deg(f)} \lambda(B(f(x), s/\eta)).$$

Thus given $\tau > 1$ there exists $\rho_0 \in (0, 1)$ such that if $x \in X$ and $0 < r \le \rho_0$, then

$$\lambda\big(B\big(x, r^{\tau/\mu(m)}\big)\big) \le \frac{m}{\deg(f)} \lambda(B(f(x), r)) \tag{11.2}$$

for some $m = m(x, r)$ satisfying $i(x, f) \le m \le \deg(f)$ .

Let $x_0 \in X$ and put $x_k := f^k(x_0)$ for $k \in \mathbb{N}$. Fix $k \in \mathbb{N}$, put $t_{k,k} := \rho_0$ and, for $0 \le j \le k - 1$, define $t_{j,k}$ recursively by $t_{j,k} := t_{j+1,k}^{\tau/\mu(m_j)}$ where $m_j := m(x_j, t_{j+1,k})$. Finally put $\rho_k := t_{0,k}$. Suppose for simplicity that $t_{j,k} \le \rho_0$ for all $j$. It then follows from (11.2) that

$$\lambda(B(x_0, \rho_k)) \le \left( \prod_{j=0}^{k-1} \frac{m_j}{\deg(f)} \right) \lambda(B(x_k, \rho_0)) \le \frac{1}{\deg(f)^k} \prod_{j=0}^{k-1} m_j.$$

Since

$$\frac{\log \rho_k}{\log \rho_0} = \prod_{j=0}^{k-1} \frac{\tau}{\mu(m_j)} = \big(\tau K_I(f)^{1/(n-1)}\big)^k \prod_{j=0}^{k-1} \left(\frac{1}{m_j}\right)^{1/(n-1)}$$

we have

$$h(\rho_k) = h(\rho_0) \left( \frac{1}{\tau^{n+\varepsilon-1} K_I(f)^{(n+\varepsilon-1)/(n-1)}} \right)^k \prod_{j=0}^{k-1} m_j^{(n+\varepsilon-1)/(n-1)}$$

for the function $h$ defined by (10.3). Choosing $\tau$ close to 1 and $\varepsilon$ small we have $\tau^{n+\varepsilon-1} K_I(f)^{(n+\varepsilon-1)/(n-1)} < \deg(f)$ and hence $\lambda(B(x_0, \rho_k)) \le h(\rho_k)$ for large $k$.

However, in order to apply Lemma 10.1 we would need that $\lambda(B(x_0, r)) \le h(r)$ for all small $r$, not only on a sequence of $r$-values. Therefore this argument can only be considered as support for Conjecture 11.1, it does not prove it.

*Remark.* With the terminology of Proposition 7.1, let $r = c\, s^{1/\mu(m)}$. It then follows from (7.3) and (7.4) that

$$f\big(\overline{B}(x,r)\big) \subset \overline{B}\big(f(x), Cr^{\mu(m)}\big) \tag{11.3}$$

and

$$n\big(\overline{B}(x,r),y\big) \le m \quad \text{for } y \in \overline{B}\big(f(x), Cr^{\mu(m)}\big) \tag{11.4}$$

for some $C>0$. However, Proposition 7.1 only says that for every $s$ there exists $m$ such that (11.3) and (11.4) hold with $r$ as defined above. It does not yield that for every $r$ there exists $m$ such that (11.3) and (11.4) hold. Under the assumption that this is the case we can prove Conjecture 11.1.

**Theorem 11.1.** *Let $f : \overline{\mathbb{R}^n} \to \overline{\mathbb{R}^n}$ be quasiregular with* $\deg(f) > K_I(f)$. *Let $X \subset \mathbb{R}^n \setminus E(f)$ be compact and completely invariant. Suppose that there exist $C, \rho > 0$ such that for every $x \in X$ and $r \in (0,\rho]$ there exists $m \in \{1,\dots,\deg(f)\}$ such that* (11.3) *and* (11.4) *hold. Then* $\operatorname{cap} X > 0$.

*Proof.* Fix $y \in X$ and let $\lambda$ be the limit of a convergent subsequence of the sequence $(\lambda_k)$ defined by (11.1). Using (11.3) and (11.4) we find that for $0 < r \le \rho$ and $x \in X$ there exists $m = m(x,r)$ such that

$$\lambda(B(x,r)) \le \frac{m}{\deg(f)} \lambda\big(B\big(f(x), Cr^{\mu(m)}\big)\big)\,.$$

Given $\alpha \in (0,1)$ there thus exists $\rho_0 > 0$ such that

$$\lambda(B(x,r)) \le \frac{m}{\deg(f)} \lambda\big(B\big(f(x), r^{\alpha\mu(m)}\big)\big)$$

for $0 < r \le \rho_0$. Now let $x \in X$ and $0 < r \le \rho_0$ and put $x_0 := x$, $r_0 := r$ and $m_0 := m(x_0, r_0)$, and define $x_k$, $r_k$ and $m_k$ for $k \ge 1$ recursively by $x_k := f(x_{k-1})$, $r_k := r_{k-1}^{\alpha\mu(m_{k-1})}$ and $m_k := m(x_k, r_k)$, as long as $r_k \le \rho_0$.

First suppose that the process stops after $k$ steps; that is, $r_k^{\alpha\mu(m_k)} > \rho_0$. Then $r_k > \rho_1 := \rho_0^{1/(\alpha\mu(m_1))}$. We conclude that

$$\frac{\log r}{\log \rho_1} \le \frac{\log r_0}{\log r_k} = \prod_{j=0}^{k-1} \alpha\mu(m_j) = \left(\frac{\alpha}{K_I(f)^{1/(n-1)}}\right)^k \prod_{j=0}^{k-1} m_j^{1/(n-1)}. \tag{11.5}$$

On the other hand,

$$\begin{aligned} \lambda(B(x,r)) &= \lambda(B(x_0,r_0)) \\ &\le \left(\prod_{j=0}^{k-1} \frac{m_j}{\deg(f)}\right) \lambda(B(x_k,r_k)) \\ &\le \frac{1}{\deg(f)^k} \prod_{j=0}^{k-1} m_j. \end{aligned} \tag{11.6}$$

Choosing $\alpha$ close to 1 we deduce from (11.5) and (11.6) that there exist $c, \varepsilon > 0$ such that

$$\lambda(B(x,r)) \le c \left(\log \frac{1}{r}\right)^{1-n-\varepsilon}. \tag{11.7}$$

Suppose now that the inductive process defining $x_k$, $r_k$ and $m_k$ does not stop; that is, $r_k \le \rho_0$ for all $k \in \mathbb{N}$. We shall show that there are infinitely many $k$ such that $m_k < \deg(f)$. In order to do so, we assume that this is not the case; say $m_k = \deg(f)$ for $k \ge k_0$. With $B_f^*$ as defined in Lemma 7.2 we then have $x_k \in B_f^*$ for $k \ge k_0$ and it follows from this lemma that there exists $\delta > 0$ such that if $|x - x_k| \le \delta$, then $|f(x) - f(x_k)| \le |x - x_k|/2$, provided $k \ge k_0$. Now there exist $l \ge k_0$ and $p \in \mathbb{N}$ such that $|x_{l+p} - x_l| \le \delta$. We deduce that $|x_{l+(j+1)p} - x_{l+jp}| \le \delta/2^{jp}$. Thus $(x_{l+jp})_{j\in\mathbb{N}}$ is a Cauchy sequence and hence convergent; say $x_{l+jp} \to \xi$. It follows that $f^p(\xi) = \xi$ and $\xi \in B_f^*$. This implies that $\xi \in E(f)$. On the other hand, we have $\xi \in X$ since $X$ is compact. Since $X \cap E(f) = \emptyset$ by hypothesis, this is a contradiction.

Thus $m_k < \deg(f)$ for infinitely many $k$. It now follows from (11.6) that $\lambda(B(x,r)) = 0$. Thus (11.7) also holds in this case. The conclusion now follows from (11.7), Lemma 9.1 and Lemma 10.1. □

*Remark.* We have restricted to sets $X \subset \mathbb{R}^n$ in Theorem 11.1 only for simplicity. It also holds for $X \subset \overline{\mathbb{R}^n}$, provided we replace the Euclidean balls in (11.3) and (11.4) by balls with respect to the chordal metric.

It follows from our considerations that if $f : \overline{\mathbb{R}^n} \to \overline{\mathbb{R}^n}$ is quasiregular with $\deg(f) > K_I(f)$ and if (11.3) and (11.4) hold (with chordal balls) for all $x \in \overline{\mathbb{R}^n}$ and $r \in (0, \rho_0]$ with some $m = m(x,r) \in \{1, \dots, \deg(f)\}$, then the conclusions of Theorems 1.5 and 1.6 hold.

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