

# ON THE UNIFORM PERFECTNESS OF THE BOUNDARY OF MULTIPLY CONNECTED WANDERING DOMAINS

WALTER BERGWELER AND JIAN-HUA ZHENG

ABSTRACT. We investigate in which cases the boundary of a multiply connected wandering domain of an entire function is uniformly perfect. We give a general criterion implying that it is not uniformly perfect. This criterion applies in particular to examples of multiply connected wandering domains given by Baker. We also provide examples of infinitely connected wandering domains whose boundary is uniformly perfect.

## 1. INTRODUCTION AND RESULTS

The *Fatou set*  $F(f)$  of an entire function  $f$  is the subset of the complex plane  $\mathbb{C}$  where the iterates  $f^n$  of  $f$  form a normal family; that is, every subsequence of  $(f^n)$  has a subsequence that converges locally uniformly. Its complement is called the *Julia set* and denoted by  $J(f)$ ; see [7] for an introduction to and discussion of these sets for transcendental functions.

The connected components of  $F(f)$  are called *Fatou components*. For a Fatou component  $U_0$  and  $k \in \mathbb{N}$  there exists a Fatou component  $U_k$  containing  $f^k(U_0)$ . A Fatou component  $U_0$  is called a *wandering domain* if  $U_j \neq U_k$  for  $j \neq k$ . While a famous theorem of Sullivan [18] says that rational functions do not have wandering domains, an example of a transcendental entire function with wandering domains had been constructed already before Sullivan's work by Baker [3]. The wandering domains in Baker's example are multiply connected. Baker [2] actually proved that multiply connected Fatou components of transcendental entire functions are always wandering.

Baker's example [3] was the function

$$(1.1) \quad f(z) = Cz^2 \prod_{k=1}^{\infty} \left(1 + \frac{z}{r_k}\right),$$

where  $C$  is a constant and  $(r_k)$  satisfies the recurrence relation

$$(1.2) \quad r_{k+1} = Cr_k^2 \prod_{j=1}^k \left(1 + \frac{r_k}{r_j}\right),$$

with  $r_1 > 1$  and  $C > 0$  chosen such that  $C \exp(2/r_1) < 1/4$  and  $Cr_1 > 1$ , for example  $C = 1/(4e)$  and  $r_1 > 4e$ . Then  $r_{k+1} \geq 2r_k$  for all  $k \in \mathbb{N}$  so that the

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1991 *Mathematics Subject Classification.* 37F10 (primary), 30D05 (secondary).

The first author is supported by a Chinese Academy of Sciences Visiting Professorship for Senior International Scientists, Grant No. 2010 TIJ10, the Deutsche Forschungsgemeinschaft, Be 1508/7-1, the EU Research Training Network CODY and the ESF Networking Programme HCAA.

The second author is supported by Grant No. 10871108 of NSF of China.

product in (1.1) converges. Baker showed that

$$f(\operatorname{ann}(0; r_k^2, \sqrt{r_{k+1}})) \subseteq \operatorname{ann}(0; r_{k+1}^2, \sqrt{r_{k+2}})$$

for large  $k$ , where  $\operatorname{ann}(a; r, R) = \{z \in \mathbb{C} : r < |z - a| < R\}$  for  $0 < r < R$  and  $a \in \mathbb{C}$ . This implies that

$$\operatorname{ann}(0; r_k^2, \sqrt{r_{k+1}}) \subseteq U_k$$

for some multiply connected Fatou component  $U_k$ . In fact, Baker had constructed the example and verified the above properties much earlier [1], but in that paper the question whether the  $U_k$  are all different had remained open. It was only in [3] that he could prove that all the  $U_k$  are different and thus wandering domains.

Many properties of this example are typical for functions with multiply connected Fatou components. We collect some of these properties in the following theorem. Here  $n(\gamma, a)$  denotes the winding number of a curve  $\gamma$  with respect to the point  $a$ .

**Theorem A.** *Let  $f$  be a transcendental entire function with a multiply connected wandering domain  $U_0$  and, for  $k \in \mathbb{N}$ , let  $U_k$  be the component of  $F(f)$  containing  $f^k(U)$ . Then we have the following:*

- (i)  $f^k|_{U_0} \rightarrow \infty$  as  $k \rightarrow \infty$ ;
- (ii) each  $U_k$  contains a closed curve  $\gamma_k$  satisfying  $n(\gamma_k, 0) \neq 0$  for large  $k$ ; in fact, if  $\gamma_0$  is a Jordan curve in  $U_0$  which is not null-homotopic, then  $\gamma_k = f^k(\gamma_0)$  has this property;
- (iii) all  $U_k$  are bounded;
- (iv) each  $U_k$  is contained in a bounded component of the complement of  $U_{k+1}$  for large  $k$ ;
- (v) there exists sequences  $(r_k)$  and  $(R_k)$  tending to  $\infty$  such that

$$\operatorname{ann}(0; r_k, R_k) \subseteq U_k$$

for large  $k$  and  $\lim_{k \rightarrow \infty} R_k/r_k = \infty$ .

Here (i) is a simple observation apparently first made by Töpfer [19, p. 67], and (ii) and (iii) are due to Baker [4, Theorem 3.1]. (He states only the first part of (ii), but his proof gives the second one.) Next, (iv) is an easy consequence of (ii). Finally, (v) was proved in [22].

Baker [5, Theorem 2] modified his construction to show that there exists an entire function  $f$  of the form

$$(1.3) \quad f(z) = C^2 \prod_{k=1}^{\infty} \left(1 + \frac{z}{r_k}\right)^2$$

which has a multiply connected Fatou component of infinite connectivity; cf. section 6. In the opposite direction, Kisaka and Shishikura [11] constructed examples where the connectivity is finite, thereby answering a question of Baker. In fact, they showed that for every  $N \in \mathbb{N}$  there exists an entire function with an  $N$ -connected Fatou component.

Next we recall that a closed subset  $K$  of  $\mathbb{C}$  is called *uniformly perfect* if there exists  $c \in (0, 1)$  such that if  $a \in K$  and  $0 < r < \operatorname{diam}(K)$ , then  $\operatorname{ann}(a; cr, r) \cap K \neq \emptyset$ . An equivalent condition is that there exists  $C > 0$  such that the modulus  $\operatorname{mod}(A)$  of any annulus  $A$  separating two components of  $K$  satisfies  $\operatorname{mod}(A) \leq C$ . Here by

an annulus we mean a doubly connected domain and the modulus of an annulus  $A$  is given by

$$\text{mod}(A) = \frac{1}{2\pi} \log \frac{R}{r},$$

where  $r$  and  $R$  with  $r < R$  are such that  $A$  can be mapped conformally onto  $\text{ann}(0; r, R)$ . Note that the ratio  $R/r$  does not depend on the conformal map. The concept of uniform perfectness was introduced by Beardon and Pommerenke [6, 14] and has found many applications in complex analysis.

It was proved by Mañé and da Rocha [12], Hinkkanen [10] and Eremenko<sup>1</sup>, with different proofs, that Julia sets of rational functions are uniformly perfect. On the other hand, the sequence  $(r_k)$  in Baker's example (1.1) satisfies  $\lim_{k \rightarrow \infty} \sqrt{r_{k+1}}/r_k^2 = \infty$  and this implies that  $J(f)$  is not uniformly perfect for this function  $f$ . In fact, it follows from Theorem A, part (v), that the Julia set of an entire function with a multiply connected Fatou component is never uniformly perfect.

Here we study the question when the boundary of a multiply connected Fatou component is uniformly perfect. Clearly, this is the case for Fatou components of finite connectivity, so it suffices to consider infinitely connected Fatou components.

For a domain  $U \subseteq \mathbb{C}$  and  $a \in \widehat{\mathbb{C}} \setminus U$  we denote by  $C(a, U)$  the component of  $\widehat{\mathbb{C}} \setminus U$  that contains  $a$ , and we put  $C(a, U) = \emptyset$  if  $a \in U$ . The union of  $U$  and its bounded complementary components is denoted by  $\widetilde{U}$ . Thus  $\widetilde{U} = \mathbb{C} \setminus C(\infty, U)$ . The set of critical points of  $f$  is denoted by  $\text{crit}(f)$ .

**Theorem 1.1.** *Let  $f$  be a transcendental entire function with a multiply connected Fatou component  $U_0$  and put  $U_k = f^k(U_0)$  for  $k \in \mathbb{N}$ . Denote by  $l_k$  the number of critical points  $c$  of  $f$  in  $\widetilde{U}_k$  for which  $f(c) \notin C(0, U_{k+1})$ , by  $m_k$  the number of zeros of  $f$  in  $C(0, U_k)$  and by  $n_k$  the number of zeros of  $f$  in  $\widetilde{U}_k$ . Suppose that*

$$(1.4) \quad l_k < m_k$$

for all large  $k$ . In addition, suppose that there exist infinitely many  $k$  such that  $U_k \cap \text{crit}(f) \neq \emptyset$  and such that  $U_{k+1}$  contains an annulus  $A_{k+1}$  which separates  $U_{k+1} \cap f(\text{crit}(f))$  from  $C(0, U_{k+1})$  and satisfies

$$(1.5) \quad \frac{\text{mod}(A_{k+1})}{n_k - m_k} \rightarrow \infty.$$

Then  $\partial U_0$  is not uniformly perfect.

It is not difficult to see that  $f$  has  $n_k - m_k$  critical points in  $\widetilde{U}_k \setminus C(0, U_k)$ . In particular, we have  $n_k > m_k$  if  $U_k$  contains a critical point. Thus the denominator in (1.5) is non-zero. Note that  $l_k \leq n_k - m_k$  so that (1.4) is satisfied in particular if

$$(1.6) \quad n_k < 2m_k.$$

We note that  $n_k - m_k$  and hence  $l_k$  is bounded in the examples (1.1) and (1.3). Thus (1.4) and (1.6) are satisfied there.

While the hypothesis of Theorem 1.1 seem somewhat complicated, the result works well for specific examples. We use Theorem 1.1 to show that in Baker's original example of a multiply connected wandering domain (i.e., the example given by (1.1) and (1.2)) the connectivity is infinite and the boundary is not uniformly perfect. The question whether the connectivity of this domain is finite or infinite

<sup>1</sup><http://www.math.purdue.edu/~eremenko/dvi/ups.pdf>, unpublished

had been raised by Baker [5] and by Kisaka and Shishikura [11]; the question was repeated in [8, p. 2946] and [15, p. 312].

More generally, we give a fairly complete discussion of functions of order 0 where the moduli  $r_k$  of the zeros satisfy a recursion formula similar to (1.2), with initial values chosen such that  $r_{k+1} \geq 2r_k$  for large  $k$ . Our result also shows that an infinitely connected wandering domain may have a uniformly perfect boundary.

**Theorem 1.2.** *Let  $(r_k)$  and  $(P_k)$  be sequences of positive numbers satisfying  $r_{k+1} \geq 2r_k$  for large  $k$ ,*

$$(1.7) \quad \lim_{k \rightarrow \infty} \sqrt[k]{P_k} = 1$$

and

$$r_{k+1} = P_k r_k^N \prod_{j=1}^k \left(1 + \frac{r_k}{r_j}\right),$$

for some non-negative integer  $N$  and all  $k$ . Let  $C \in \mathbb{C} \setminus \{0\}$ , let  $(a_k)$  be a sequence of complex numbers satisfying  $|a_k| = r_k$  and define the entire function  $f$  by

$$(1.8) \quad f(z) = Cz^N \prod_{k=1}^{\infty} \left(1 - \frac{z}{a_k}\right).$$

Then there exist  $K \in \mathbb{N}$  and a sequence  $(\varepsilon_k)$  of positive real numbers tending to 0 such that

$$(1.9) \quad \begin{aligned} & f(\text{ann}(0; (1 + \varepsilon_k)r_k, (1 - \varepsilon_k)r_{k+1})) \\ & \subseteq \text{ann}(0; (1 + \varepsilon_{k+1})r_{k+1}, (1 - \varepsilon_{k+1})r_{k+2}) \end{aligned}$$

for  $k \geq K$ .

Denote, for  $k \geq K$ , the Fatou component containing

$$\text{ann}(0; (1 + \varepsilon_k)r_k, (1 - \varepsilon_k)r_{k+1})$$

by  $U_k$ . If

$$\limsup_{k \rightarrow \infty} kP_k > \frac{|C|}{2e} \quad \text{or} \quad \liminf_{k \rightarrow \infty} kP_k < \frac{|C|}{2e}$$

then  $U_k$  is infinitely connected for all  $k$ , and if

$$\limsup_{k \rightarrow \infty} kP_k > \frac{|C|}{2e} \quad \text{or} \quad \liminf_{k \rightarrow \infty} kP_k = 0,$$

then  $\partial U_k$  is not uniformly perfect for all  $k$ . If

$$(1.10) \quad \limsup_{k \rightarrow \infty} kP_k < \frac{|C|}{2e} \quad \text{and} \quad \liminf_{k \rightarrow \infty} kP_k > 0,$$

then  $\partial U_k$  is uniformly perfect for all  $k$ .

Noting that  $P_k = C$  and thus  $\lim_{k \rightarrow \infty} kP_k = \infty$  in Baker's first example of a wandering domain, we deduce that this wandering domain is infinitely connected and that its boundary is not uniformly perfect.

In principle a similar discussion could be done for functions of the form

$$f(z) = Cz^N \prod_{k=1}^{\infty} \left(1 - \frac{z}{a_k}\right)^2,$$

but here we will only prove that in Baker's example (1.3) of an infinitely connected wandering domain the boundary is also not uniformly perfect; see section 6.

In the functions considered in Theorem 1.2, as well as in Baker's example (1.3), we have  $n_k - m_k \leq 2$  and thus the condition (1.5) just says that  $\text{mod}(A_{k+1}) \rightarrow \infty$ . In the following example we have  $n_k - m_k \rightarrow \infty$ . The example shows that (1.5) is best possible in some sense.

**Theorem 1.3.** *Let  $q_0$  be an even integer and put  $a_0 = \exp(q_0/2)$ . Define sequences  $(q_k)$  and  $(a_k)$  recursively by*

$$(1.11) \quad q_{k+1} = \frac{3}{2}q_k^2 \quad \text{and} \quad a_{k+1} = \exp q_k.$$

Then

$$f(z) = z^2 \prod_{k=0}^{\infty} \left(1 - \frac{z}{a_k}\right)^{q_k}$$

defines an entire function  $f$  and if  $q_0$  is sufficiently large, then  $f$  has an infinitely connected Fatou component  $U_0$  whose boundary is uniformly perfect.

Moreover, (1.4) is satisfied, each  $U_k$  contains exactly one critical point  $c_k$  and  $U_{k+1}$  contains an annulus  $A_{k+1}$  separating  $f(c_k)$  from  $C(0, U_{k+1})$  such that

$$(1.12) \quad \frac{\text{mod}(A_{k+1})}{n_k - m_k} \geq c$$

for some positive constant  $c$  and all  $k$ .

There is an interesting difference between the example given by Theorem 1.3 and the examples obtained from Theorem 1.2 by choosing  $(P_k)$  such that (1.10) holds. Our proofs will show that in the example given by Theorem 1.3 the complementary components of the wandering domain cluster only at the "outer boundary" while in the examples obtained from Theorem 1.2 they cluster at the "inner boundary". We discuss this in more detail in a remark at the end of section 5.

**Acknowledgement.** *We thank the referee for many helpful comments and Alexandre Eremenko for a discussion of his unpublished preprint and references [10, 12].*

## 2. PRELIMINARIES

We denote the connectivity of a domain  $G$  in  $\mathbb{C}$  by  $\text{conn}(G)$ ; that is,  $\text{conn}(G)$  is the number of connected components of  $\widehat{\mathbb{C}} \setminus G$ . The following result is known as the Riemann-Hurwitz formula; see, e.g., [16, p. 7].

**Lemma 2.1.** *Let  $G$  and  $H$  be domains in  $\mathbb{C}$  and let  $f : G \rightarrow H$  be a proper holomorphic map of degree  $d$  with  $m$  critical points, counting multiplicity. Then*

$$\text{conn}(G) - 2 = d(\text{conn}(H) - 2) + m.$$

Here it is understood that if one of the domains is of infinite connectivity, then so is the other one.

It follows from Theorem A, part (iii), that if  $f$  is an entire function with a multiply connected wandering domain  $U_0$  and if  $U_k$  is the component of  $F(f)$  containing  $f^k(U)$ , then  $f : U_k \rightarrow U_{k+1}$  is a proper map. In particular,  $U_k = f^k(U_0)$ .

The following consequence of the Riemann-Hurwitz formula can be found in [5, Lemma 6] and [11, Theorem A].

**Lemma 2.2.** *Let  $f$  be a transcendental entire function with a multiply connected wandering domain  $U_0$ . If  $\bigcup_{k=0}^{\infty} f^k(U)$  contains infinitely many critical points, then  $U_0$  is infinitely connected.*

It was shown in [9, Theorem 1.7] that the converse also holds: if  $\bigcup_{k=0}^{\infty} U_k$  contains only finitely many critical points, then  $U_0$  has finite connectivity. We will not need this result, but we mention that it shows that the hypothesis in Theorem 1.1 that infinitely many  $U_k$  contain a critical point is automatically fulfilled if  $U_0$  is infinitely connected.

**Lemma 2.3.** *Let  $G$  and  $H$  be simply-connected domains, let  $f : G \rightarrow H$  be a proper holomorphic map of degree  $d$  and let  $A$  be an annulus in  $H$ . Denote by  $p$  the number of critical points  $c \in G$  for which  $f(c) \in \tilde{A}$ , counted according to multiplicity. If  $d > 2p$ , then  $f^{-1}(A)$  has a component  $B$  such that  $f : B \rightarrow A$  is univalent. In particular,  $\text{mod}(B) = \text{mod}(A)$ .*

**Proof.** Let  $C_1, \dots, C_k$  be the components of  $f^{-1}(\tilde{A})$  and let  $r_j$  be the number of critical points in  $C_j$ . Then  $\sum_{j=1}^k r_j = p$ . Now  $f : C_j \rightarrow \tilde{A}$  is a proper map and its degree is  $r_j + 1$  by the Riemann-Hurwitz formula, since  $C_j$  and  $\tilde{A}$  are simply connected. Thus

$$d = \sum_{j=1}^k (r_j + 1) = k + \sum_{j=1}^k r_j = k + p.$$

By hypothesis, we have  $d > 2p$  and this implies that  $k > p$ . Thus there exists  $j$  such that  $C_j$  contains no critical point. Hence  $f : C_j \rightarrow \tilde{A}$  is univalent and the conclusion follows.  $\square$

We denote the density of the hyperbolic metric in a hyperbolic domain  $U$  by  $\varrho_U$  and the hyperbolic length of a curve  $\gamma$  in  $U$  by  $\text{length}(\gamma, U)$ . Thus  $\text{length}(\gamma, U) = \int_{\gamma} \varrho_U(z) |dz|$ . The following result is well-known; see [17, Theorem 2.3] or [21, Proposition 3].

**Lemma 2.4.** *Let  $U$  be a hyperbolic domain. Then  $\partial U$  is uniformly perfect if and only if there exists  $\delta > 0$  such that  $\text{length}(\gamma, U) \geq \delta$  for each curve  $\gamma$  in  $U$  which is not null-homotopic.*

The next lemma is also standard, but for convenience we include the proof. Here  $n(\gamma, a)$  denotes the winding number of a curve  $\gamma$  with respect to a point  $a$ .

**Lemma 2.5.** *Let  $0 < r < R$  and let  $\gamma$  be a closed curve in  $\text{ann}(0; r, R)$ . Then*

$$\text{length}(\gamma, \text{ann}(0; r, R)) \geq \frac{2\pi^2 |n(\gamma, 0)|}{\log(R/r)}.$$

**Proof.** We may assume that  $r = 1$ . The density of the hyperbolic metric in  $\text{ann}(0; 1, R)$  is given by (see, e.g., [13, p. 12])

$$\varrho_{\text{ann}(0; 1, R)}(z) = \frac{\pi}{|z| \sin(\pi \log |z| / \log R) \log R}.$$

In particular, we have

$$\varrho_{\text{ann}(0; 1, R)}(z) \geq \frac{\pi}{|z| \log R}$$

and thus

$$\begin{aligned}
\text{length}(\gamma, \text{ann}(0; 1, R)) &= \int_{\gamma} \varrho_{\text{ann}(0; 1, R)}(z) |dz| \\
&\geq \frac{\pi}{\log R} \int_{\gamma} \frac{|dz|}{|z|} \\
&\geq \frac{\pi}{\log R} \int_{\gamma} |d \arg z| \\
&\geq \frac{\pi}{\log R} \left| \int_{\gamma} d \arg z \right| \\
&= \frac{2\pi^2}{\log R} |n(\gamma, 0)|.
\end{aligned}$$

□

The following result can be found in [20, Theorem 3].

**Lemma 2.6.** *Let  $U, V$  be domains in  $\mathbb{C}$  and let  $f : U \rightarrow V$  be a proper holomorphic map. Then  $\partial U$  is uniformly perfect if and only if  $\partial V$  is uniformly perfect.*

Lemma 2.6 implies that if an entire function  $f$  has a multiply connected wandering domain  $U_0$  and  $U_k$  is the Fatou component containing  $f^k(U)$  so that  $U_k = f^k(U)$  as remarked after Lemma 2.1, then  $\partial U_0$  is uniformly perfect if and only if  $\partial U_k$  is uniformly perfect.

### 3. PROOF OF THEOREM 1.1

It follows from Theorem A, part (ii), that

$$(3.1) \quad 0 \notin U_k \quad \text{and} \quad 0 \in \widetilde{U}_k$$

for large  $k$ . In view of Lemma 2.6 and the remark following it, we may assume that (3.1) holds for all  $k \geq 0$ .

Now let  $k$  be an index such that  $U_k$  contains a critical point  $c_k$  and let  $A_{k+1}$  be an annulus as given in the hypothesis. Choosing  $A_{k+1}$  slightly smaller if necessary, we may assume that  $f(c) \notin \overline{A_{k+1}}$  for every critical point  $c \in U_k$  and that  $\overline{A_{k+1}} \subset U_{k+1}$ . Thus there exists an annulus  $B_{k+1} \subset U_{k+1}$  separating  $A_{k+1}$  and  $C(\infty, U_{k+1})$  such that  $f(c_k) \in B_{k+1}$ . We may also assume that  $A_{k+1}$  and  $B_{k+1}$  are bounded by smooth curves.

Let  $V_k$  be the component of  $f^{-1}(B_{k+1}) \cap U_k$  that contains  $c_k$ . By the Riemann-Hurwitz formula,  $V_k$  is at least triply connected. Thus there exists a component  $X_k$  of  $\widehat{\mathbb{C}} \setminus V_k$  satisfying  $X_k \neq C(0, V_k)$  and  $X_k \neq C(\infty, V_k)$ . Thus  $0 \notin X_k$  and  $X_k \subset \widetilde{U}_k$ . Moreover,  $X_k \cap U_k$  contains a component  $Y_k$  of  $f^{-1}(A_{k+1})$ . Since  $f(c) \notin A_{k+1}$  for every critical point  $c \in U_k$ , we find that  $Y_k$  is also an annulus and  $f : Y_k \rightarrow A_{k+1}$  is a covering. Moreover,  $f : \widetilde{Y}_k \rightarrow \widetilde{A_{k+1}}$  is a proper map and its degree  $d_k$  equals the number of zeros of  $f$  in  $\widetilde{Y}_k$ . Since  $\widetilde{Y}_k \subset X_k \subset \widetilde{U}_k \setminus C(0, U_k)$  we have  $d_k \leq n_k - m_k$ . On the other hand,  $d_k$  is equal to the degree of the covering  $f : Y_k \rightarrow A_{k+1}$ . We thus have

$$\text{mod}(Y_k) = \frac{\text{mod}(A_{k+1})}{d_k} \geq \frac{\text{mod}(A_{k+1})}{n_k - m_k}.$$

Hence  $U_k$  contains an annulus  $Y_k$  with  $0 \notin \widetilde{Y}_k$  and  $\text{mod}(Y_k) \rightarrow \infty$  as  $k \rightarrow \infty$ .

Now let  $p_{k-1}$  be the number of critical points  $c$  of  $f$  in  $\widetilde{U_{k-1}}$  for which  $f(c) \in \widetilde{Y_k}$ . Since  $\widetilde{Y_k} \subset X_k$  and  $X_k \neq C(0, U_k)$  we have  $p_{k-1} \leq l_{k-1}$ . Without loss of generality we may assume that (1.4) holds for all  $k$  and thus we have  $p_{k-1} < m_{k-1}$ . Since  $n_{k-1}$  is the degree of the proper map  $f : \widetilde{U_{k-1}} \rightarrow \widetilde{U_k}$  we have  $n_{k-1} \geq m_{k-1} + l_{k-1} > 2p_{k-1}$ . We deduce from Lemma 2.3 that there exists an annulus  $Y_k^1 \subset U_{k-1}$  which is mapped univalently onto  $Y_k$  by  $f$ . Applying Lemma 2.3 again we find an annulus  $Y_k^2 \subset U_{k-2}$  which is mapped univalently onto  $Y_k$  by  $f^2$ . Inductively we thus obtain a non-trivial annulus  $Y_k^k \subset U_0$  which is mapped univalently onto  $Y_k$  by  $f^k$ . Since  $\text{mod}(Y_k^k) = \text{mod}(Y_k) \rightarrow \infty$  as  $k \rightarrow \infty$  by our hypothesis (1.5), we conclude that  $\partial U_0$  is not uniformly perfect.

#### 4. PROOF OF THEOREM 1.2

Since  $r_{k+1} \geq 2r_k$  for large  $k$  we easily see that the product defining  $f$  converges. We have

$$r_{k+1} = P_k r_k^N \prod_{j=1}^k \left(1 + \frac{r_k}{r_j}\right) \geq P_k \frac{r_k}{r_1} \prod_{j=2}^{k-1} \frac{r_k}{r_j} \geq P_k \frac{r_k}{r_1} \left(\frac{r_k}{r_{k-1}}\right)^{k-2}$$

for  $k \geq 2$ . Together with (1.7) this implies that not only  $r_{k+1} \geq 2r_k$ , but

$$\lim_{k \rightarrow \infty} \frac{r_{k+1}}{r_k} = \infty.$$

Moreover, we have

$$(4.1) \quad \frac{r_{k+1}}{r_k} \geq \frac{P_k}{r_1} \left(\frac{r_k}{r_{k-1}}\right)^{k-2} \geq \frac{P_k}{r_1} 3^{k-k_0} \geq 2^k$$

for some  $k_0$  and large  $k \geq k_0$ . This implies that if  $w \in \mathbb{C} \setminus \{0\}$ , then

$$(4.2) \quad \begin{aligned} \left| \prod_{j=1}^{k-1} \left(1 - \frac{wa_k}{a_j}\right) \right| &\sim \left| w^{k-1} \prod_{j=1}^{k-1} \frac{a_k}{a_j} \right| \\ &= |w|^{k-1} \prod_{j=1}^{k-1} \frac{r_k}{r_j} \\ &\sim |w|^{k-1} \prod_{j=1}^{k-1} \left(1 + \frac{r_k}{r_j}\right) \end{aligned}$$

and

$$(4.3) \quad \left| \prod_{j=k+1}^{\infty} \left(1 - \frac{wa_k}{a_j}\right) \right| \rightarrow 1$$



as  $k \rightarrow \infty$ . Given  $\beta > \alpha > 0$ , we actually find that (4.2) and (4.3) hold uniformly for  $\alpha \leq |w| \leq \beta$ . It follows that

$$\begin{aligned}
|f(wa_k)| &\sim \left| C (wa_k)^N \right| |w|^{k-1} \prod_{j=1}^{k-1} \left( 1 + \frac{r_k}{r_j} \right) |w-1| \\
(4.4) \qquad &= |C| |w|^{k-1+N} r_k^N \prod_{j=1}^{k-1} \left( 1 + \frac{r_k}{r_j} \right) |w-1| \\
&= |C| \frac{|w|^{k-1+N} |w-1|}{2P_k} r_{k+1}
\end{aligned}$$

as  $k \rightarrow \infty$ , uniformly for  $\alpha \leq |w| \leq \beta$ .

Fix  $\varepsilon \in (0, 1/3]$ . For  $|z| = (1 + \varepsilon)r_k = (1 + \varepsilon)|a_k|$  we deduce from (4.4) that

$$|f(z)| \geq (1 - o(1)) \frac{|C|(1 + \varepsilon)^{k-1+N} \varepsilon}{2P_k} r_{k+1}$$

and thus

$$(4.5) \qquad |f(z)| \geq 2r_{k+1} \quad \text{for } |z| = (1 + \varepsilon)r_k$$

if  $k$  is sufficiently large. Similarly we find that

$$(4.6) \qquad |f(z)| \leq \frac{1}{2} r_{k+1} \quad \text{for } |z| = (1 - \varepsilon)r_k$$

for large  $k$ . Moreover, it follows from the above reasoning and (4.1) that if  $|z| = (1 - \varepsilon)r_k$ , then

$$\begin{aligned}
|f(z)| &\geq (1 - o(1)) \frac{|C|(1 - \varepsilon)^{k-1+N} \varepsilon}{2P_k} r_{k+1} \\
&\geq (1 - o(1)) \frac{|C|(1 - \varepsilon)^{k-1+N} \varepsilon 2^k}{2P_k} r_k.
\end{aligned}$$

Thus

$$(4.7) \qquad |f(z)| \geq 2r_k \quad \text{for } |z| = (1 - \varepsilon)r_k$$

for large  $k$ .

Finally, replacing  $k$  by  $k + 1$  in (4.6) we conclude that  $|f(z)| \leq r_{k+2}/2$  for  $|z| = (1 - \varepsilon)r_{k+1}$ . Now  $(1 - \varepsilon)r_{k+1} \geq (1 + \varepsilon)r_k$  since  $r_{k+1} \geq 2r_k$  and  $0 < \varepsilon \leq 1/3$ . Thus, by the maximum principle, the last inequality also holds for  $|z| = (1 + \varepsilon)r_k$ . Hence

$$(4.8) \qquad |f(z)| \leq \frac{1}{2} r_{k+2} \quad \text{for } |z| = (1 + \varepsilon)r_k.$$

Now (1.9) follows from (4.5)–(4.8) and the maximum and minimum principle.

Next we show that for large  $k$  the function  $f$  has exactly one critical point in the closed annulus

$$B_k = \overline{\text{ann} \left( 0; \sqrt{r_k r_{k-1}}, \sqrt{r_{k+1} r_k} \right)}$$

and that if we denote this critical point by  $c_k$ , then

$$(4.9) \qquad c_k = \left( 1 - \frac{1}{k + N + \delta_k} \right) a_k$$

for some sequence  $(\delta_k)$  tending to 0. In order to do this we note that if  $z \in B_k$ , then

$$\begin{aligned}
(4.10) \quad & \left| \frac{f'(z)}{f(z)} - \frac{k-1+N}{z} - \frac{1}{z-a_k} \right| \\
&= \left| \sum_{j=1}^{k-1} \left( \frac{1}{z-a_j} - \frac{1}{z} \right) + \sum_{j=k+1}^{\infty} \frac{1}{z-a_j} \right| \\
&\leq \sum_{j=1}^{k-1} \frac{r_j}{|z|(|z|-r_j)} + \sum_{j=k+1}^{\infty} \frac{1}{r_j-|z|} \leq \sum_{j=1}^{k-1} \frac{2r_j}{|z|^2} + \sum_{j=k+1}^{\infty} \frac{2}{r_j} \\
&\leq \frac{4r_{k-1}}{|z|^2} + \frac{4}{r_{k+1}} \leq 4 \left( \sqrt{\frac{r_{k-1}}{r_k}} + \sqrt{\frac{r_k}{r_{k+1}}} \right) \frac{1}{|z|} = o\left(\frac{1}{|z|}\right)
\end{aligned}$$

as  $k \rightarrow \infty$ .

Using Rouché's theorem we deduce from (4.10) that the difference between the number of zeros and poles in  $B_k$  is the same for  $f'/f$  and the function given by

$$z \mapsto \frac{k-1+N}{z} + \frac{1}{z-a_k},$$

provided  $k$  is sufficiently large. We conclude that if  $k$  is large, then  $f'$  has exactly one zero  $c_k \in B_k$ . Moreover, (4.10) implies that

$$\frac{k-1+N}{c_k} + \frac{1}{c_k-a_k} = o\left(\frac{1}{|c_k|}\right)$$

as  $k \rightarrow \infty$ . This yields (4.9).

It follows from (4.4) and (4.9) that

$$(4.11) \quad |f(c_k)| \sim |C| \frac{\left|1 - \frac{1}{k+N+\delta_k}\right|^{k-1+N}}{2P_k} \left|\frac{1}{k+N+\delta_k}\right| r_{k+1} \sim \frac{|C|}{2ekP_k} r_{k+1}$$

as  $k \rightarrow \infty$ .

Suppose now that  $\limsup_{k \rightarrow \infty} kP_k > |C|/(2e)$ . Then there exists  $\varepsilon > 0$  such that

$$|f(c_k)| < (1-\varepsilon)r_{k+1}$$

for infinitely many  $k$ . On the other hand, we have

$$|f(c_k)| > (1-\varepsilon) \frac{|C|}{2ekP_k} r_{k+1} \geq (1-\varepsilon) \frac{|C|3^{k-k_0}}{2ekr_1} r_k \geq kr_k$$

for large  $k$  by (4.1) and (4.11). Thus  $f(c_k) \in U_k$  for infinitely many  $k$  and Lemma 2.2 implies that all  $U_k$  are infinitely connected.

Moreover, noting that  $n_{k-1} - m_{k-1} \leq 2$  we see that (1.5), with  $k$  replaced by  $k-1$ , holds for

$$A_k = \text{ann}(0; (1+\varepsilon_k)r_k, kr_k) \subseteq U_k.$$

Clearly, (1.6) and hence (1.4) are also satisfied. Theorem 1.1 yields that  $\partial U_k$  is not uniformly perfect for  $k \geq K$ .

Similarly, if  $\liminf_{k \rightarrow \infty} kP_k < |C|/(2e)$ , then

$$(4.12) \quad (1+\varepsilon)r_{k+1} < |f(c_k)| < 2^k r_{k+1} \leq \frac{1}{2} r_{k+2}$$

and hence  $f(c_k) \in U_{k+1}$  for infinitely many  $k$  by (1.7), (4.1) and (4.11). Again we deduce from Lemma 2.2 that all  $U_k$  are infinitely connected. Moreover, if  $\liminf_{k \rightarrow \infty} kP_k = 0$ , then  $|f(c_k)|/r_{k+1} \rightarrow \infty$  and with

$$A_{k+1} = \text{ann}(0; (1 + \varepsilon_{k+1})r_{k+1}, |f(c_k)|)$$

we deduce from Theorem 1.1 that  $\partial U_k$  is not uniformly perfect for  $k \geq K$ .

Suppose now that (1.10) holds. We show first that if  $k$  is large, then

$$(4.13) \quad \left\{ z \in \mathbb{C} : \left(1 - \frac{1}{k}\right)r_k \leq |z| < (1 - \varepsilon_k)r_{k+1}, |z - a_k| \geq \frac{1}{k}r_k \right\} \subseteq U_k,$$

but

$$(4.14) \quad U_k \cap D\left(a_k, \frac{\delta}{k}r_k\right) = \emptyset$$

and

$$(4.15) \quad U_k \cap \left[\frac{1}{2}a_k, \left(1 - \frac{\tau}{k}\right)a_k\right] = \emptyset$$

for certain  $\tau, \delta > 0$ . (Here  $[u, v] = \{u + t(v - u) : 0 \leq t \leq 1\}$  is the line segment connecting two points  $u, v \in \mathbb{C}$ .)

It follows from (4.6) and (4.7) that in order to prove (4.13) it suffices to show that there exists  $\varepsilon > 0$  such that

$$(4.16) \quad |f(z)| \geq (1 + \varepsilon)r_{k+1} \quad \text{for} \quad |z| = \left(1 - \frac{1}{k}\right)r_k$$

and

$$(4.17) \quad |f(z)| \geq (1 + \varepsilon)r_{k+1} \quad \text{for} \quad |z - a_k| = \frac{1}{k}r_k$$

for large  $k$ . Now (4.4) yields

$$|f(z)| \geq (1 - o(1))|C| \frac{\left(1 - \frac{1}{k}\right)^{k-1+N}}{2kP_k} r_{k+1} = (1 - o(1)) \frac{|C|}{2ekP_k} r_{k+1}$$

for  $|z| = \left(1 - \frac{1}{k}\right)r_k$ . Since  $\limsup_{k \rightarrow \infty} kP_k < |C|/(2e)$  by our assumption, (4.16) follows. Essentially the same argument also yields (4.17) and thus we obtain (4.13).

On the other hand, if  $|z - a_k| = \delta r_k/k$ , then  $|z| \leq (1 + \delta/k)r_k$  and we find by similar estimates as before that

$$|f(z)| \leq (1 + o(1)) \frac{|C| \left(1 + \frac{\delta}{k}\right)^{k-1+N} \delta}{2kP_k} r_{k+1} = (1 + o(1)) \frac{|C|e^\delta \delta}{2kP_k} r_{k+1}.$$

Since we assumed that  $\liminf_{k \rightarrow \infty} kP_k > 0$  we see that if  $\delta$  is sufficiently small, then

$$(4.18) \quad |f(z)| \leq \frac{1}{2}r_{k+1}$$

for large  $k$ . Thus  $f(z) \in \widetilde{U}_k$ . Since  $\widetilde{U}_k \subset C(0, U_{k+1})$  and thus

$$\widetilde{U}_k \cap U_{k+1} = \widetilde{U}_k \cap f(U_k) = \emptyset,$$

this yields (4.14).

Also, if  $z = sa_k$  where  $1/2 \leq s \leq (1 - \tau/k)$ , then

$$|f(z)| \sim \frac{|C|s^{k-1+N}(1-s)}{2P_k}$$

by (4.4). For  $\tau \geq 2$  the function  $s \mapsto s^{k-1+N}(1-s)$  is increasing in the interval  $[1/2, (1-\tau/k)]$  and thus we obtain

$$|f(z)| \leq (1+o(1)) \frac{|C| \left(1 - \frac{\tau}{k}\right)^{k-1+N} \tau}{2kP_k} r_{k+1} \sim \frac{|C| e^{-\tau} \tau}{2kP_k} r_{k+1}.$$

If  $\tau$  is chosen sufficiently large, our assumption that  $\liminf_{k \rightarrow \infty} kP_k > 0$  implies (4.18) for large  $k$ . The same argument as above now yields (4.15).

We also note that by construction we have

$$(4.19) \quad U_k \subseteq \text{ann}(0; (1 - \varepsilon_{k-1})r_k, (1 + \varepsilon_{k+1})r_{k+1}) \subseteq \text{ann}(0; 1, 2r_{k+1})$$

for large  $k$ . We may assume that (4.13), (4.14), (4.15) and (4.19) hold for  $k \geq K$ .

In order to prove that  $\partial U_K$  is uniformly perfect we use Lemma 2.4. Let  $\sigma_K$  be a curve in  $U_K$  which is not null-homotopic. If  $n(\sigma_K, 0) \neq 0$ , then

$$(4.20) \quad \text{length}(\sigma_K, U_K) \geq \text{length}(\sigma_K, \text{ann}(0; 1, 2r_{K+1})) \geq \frac{2\pi^2}{\log(2r_{K+1})}$$

by (4.19) and Lemma 2.5. Suppose that  $n(\sigma_K, 0) = 0$ . Put  $\sigma_k = f^{k-K}(\sigma_K)$  for  $k > K$ . By Theorem A, part (ii), we have  $n(\sigma_k, 0) \neq 0$  for large  $k$ . Thus there exists  $k \geq K$  such that  $n(\sigma_k, 0) = 0$  and  $n(\sigma_{k+1}, 0) \neq 0$ . It follows that  $n(\sigma_k, a) \neq 0$  for some zero  $a$  of  $f$ , and (4.13) implies that we actually have  $n(\sigma_k, a_k) \neq 0$ . Let

$$V_k = \mathbb{C} \setminus \left( \overline{D\left(a_k, \frac{\delta}{k}r_k\right)} \cup \left[ \frac{1}{2}a_k, \left(1 - \frac{\tau}{k}\right)a_k \right] \right).$$

Since  $U_k \subseteq V_k$  by (4.14) and (4.15), we have

$$(4.21) \quad \text{length}(\sigma_K, U_K) \geq \text{length}(\sigma_k, U_k) \geq \text{length}(\sigma_k, V_k).$$

We put

$$T_k(z) = k \frac{z - a_k}{a_k} \quad \text{and} \quad W_k = T_k(V_k) = \mathbb{C} \setminus \left( \overline{D(0, \delta)} \cup \left[ -\frac{1}{2}k, -\tau \right] \right).$$

We may assume that  $K \geq 4\tau$ . With

$$W = \mathbb{C} \setminus \left( \overline{D(0, \delta)} \cup [-2\tau, -\tau] \right)$$

we then have  $W_k \subseteq W$  for  $k \geq K$ . Since  $a_k/2 \in C(0, U_k)$  we  $n(\sigma_k, a_k/2) = 0$ . On the other hand,  $n(\sigma_k, a_k) \neq 0$  and thus  $\sigma_k$  separates  $[a_k/2, (1-\tau/k)a_k]$  and  $\overline{D(a_k, \delta r_k/k)}$ . Hence  $T_k(\sigma_k)$  is a curve in  $W$  which separates  $\overline{D(0, \delta)}$  and  $[-2\tau, -\tau]$ . Thus

$$(4.22) \quad \text{length}(\sigma_k, V_k) = \text{length}(T_k(\sigma_k), W_k) \geq \text{length}(T_k(\sigma_k), W) \geq c$$

for some positive constant  $c$ . Combining (4.21) and (4.22) we obtain

$$(4.23) \quad \text{length}(\sigma_K, U_K) \geq c.$$

Now (4.20) and (4.23) imply together with Lemma 2.4 that  $\partial U_K$  is uniformly perfect. Thus  $\partial U_k$  is uniformly perfect for all  $k \geq K$  by Lemma 2.6

**Remark 4.1.** *As mentioned in the introduction, Kisaka and Shishikura [11] constructed an example of an entire function  $f$  with a doubly connected wandering domain. In order to ensure that the wandering domains do not contain critical points, which has to be avoided by Lemma 2.2, they construct it in such a way that  $f(0) = 0$  and  $f^2(c) = 0$  for each critical point  $c$  of  $f$ . Their construction uses*

quasiconformal surgery, but it turns out that the function obtained is of order zero and can be written in the form (1.8) with a sequence  $(a_k)$  which tends to  $\infty$  rapidly. The sequence  $(c_k)$  of critical points again satisfies (4.9) and the construction is such that  $f(c_k) = a_{k+1}$  and thus  $f^2(c_k) = 0$ . It follows from the arguments in the above proof that

$$a_{k+1} = f(c_k) \sim a_k^N \frac{C}{ek} \prod_{j=0}^{k-1} \frac{a_k}{a_j}$$

as  $k \rightarrow \infty$ .

### 5. PROOF OF THEOREM 1.3

It follows easily from Rolle's theorem that for a real polynomial with real zeros, each open interval on the real axis bounded by two adjacent zeros contains exactly one critical point, this critical point is simple, and there are no further critical points except for multiple zeros. Since  $f$  is a locally uniform limit of polynomials with real zeros, the above result also holds for  $f$ . We thus find that for  $k \geq 0$  there exists a critical point  $c_k \in (a_k, a_{k+1})$  and except 0 and one further critical point in the interval  $(0, a_0)$  there are no critical points other than the  $a_k$  and  $c_k$ .

We will show that if  $q_0$  is sufficiently large, then the following properties are satisfied for all  $k \geq 0$ :

$$(5.1) \quad f(\text{ann}(4a_k, a_{k+1}/4)) \subseteq \text{ann}(4a_{k+1}, a_{k+2}/4),$$

$$(5.2) \quad f(\text{ann}(4a_k, \sqrt{a_{k+1}})) \subseteq \text{ann}(4a_{k+1}, \sqrt{a_{k+2}}),$$

$$(5.3) \quad \frac{q_k}{2q_{k+1}} a_{k+1} < c_k < \frac{2q_k}{q_{k+1}} a_{k+1},$$

and

$$(5.4) \quad \sqrt{a_{k+2}} < f(c_k) < \frac{1}{4} a_{k+2}.$$

Suppose that (5.1)–(5.4) hold. Then  $f$  has a wandering domain  $U_0$  containing  $\text{ann}(4a_0, a_1/4)$ .

We consider the annulus  $X_k = D(0, \sqrt{a_{k+1}}) \setminus C(0, U_k)$ . If  $k \geq 1$ , then  $X_k$  contains no critical values by (5.4) and thus the components of  $f^{-1}(X_k)$  are also annuli. In particular, this holds for the component  $Y_{k-1}$  of  $f^{-1}(X_k)$  whose boundary intersects  $C(0, U_{k-1})$ . By (5.2) we have  $Y_{k-1} \supseteq X_{k-1}$  and thus  $f(X_{k-1}) \subseteq f(Y_{k-1}) = X_k$ . It follows that  $X_k \subseteq F(f)$  and thus  $X_k \subseteq U_k$  for all  $k \geq 0$ . This implies that

$$(5.5) \quad a_j \in C(0, U_k) \quad \text{for } 0 \leq j \leq k.$$

We also note that

$$(5.6) \quad U_k \subseteq \text{ann}(a_k/4, 4a_{k+1}) \subseteq \text{ann}(1, 4a_{k+1})$$

for  $k \geq 1$  and

$$(5.7) \quad U_0 \subseteq \text{ann}(\delta, 4a_1)$$

for some  $\delta > 0$ , since 0 is a superattracting fixed point.

As in the proof of Theorem 1.2 we use Lemma 2.4 to show that  $\partial U_0$  is uniformly perfect. So let  $\sigma_0$  be a Jordan curve in  $U_0$  which is not null-homotopic. If  $n(\sigma_0, 0) \neq 0$ , then

$$(5.8) \quad \text{length}(\sigma_0, U_0) \geq \frac{2\pi^2}{\log(4a_1/\delta)}$$

by (5.7) and Lemma 2.5.

We now assume that  $n(\sigma_0, 0) = 0$ . Put  $\sigma_k = f^k(\sigma_0)$ . By Theorem A, part (ii), we have  $n(\sigma_k, 0) \neq 0$  for large  $k$  and thus there exists  $k \geq 1$  such that  $n(\sigma_k, 0) \neq 0$  while  $n(\sigma_{k-1}, 0) = 0$ . It follows that  $n(\sigma_{k-1}, a) \neq 0$  for some zero  $a$  of  $f$ . Using (5.5) we see that  $n(\sigma_{k-1}, a_k) \neq 0$  while  $n(\sigma_{k-1}, a_j) = 0$  for  $j \neq k$ . Since  $a_k$  is a zero of multiplicity  $q_k$  this implies that

$$n(\sigma_k, 0) = n(f(\sigma_{k-1}), 0) = q_k n(\sigma_{k-1}, a_k).$$

In particular,  $|n(\sigma_k, 0)| \geq q_k$ . Combining this with (5.6) and Lemma 2.5 we obtain

$$\text{length}(\sigma_k, U_k) \geq \text{length}(\sigma_k, \text{ann}(1, 4a_{k+1})) \geq \frac{2\pi^2 q_k}{\log(4a_{k+1})} = \frac{2\pi^2 q_k}{q_k + \log 4} \geq \pi^2.$$

On the other hand,  $\text{length}(\sigma_k, U_k) \leq \text{length}(\sigma_0, U_0)$  and thus we obtain

$$(5.9) \quad \text{length}(\sigma_0, U_0) \geq \pi^2$$

in this case. Thus for each Jordan curve  $\sigma_0$  in  $U_0$  which is not null-homotopic we have (5.8) or (5.9). Lemma 2.4 now implies that  $\partial U_0$  is uniformly perfect.

Next we note that  $l_k = 1$  since  $f(c_j) \in U_{j+1} \subset C(0, U_{k+1})$  for  $j < k$  by (5.4) so that  $c_k$  is the only critical point of  $f$  in  $\widetilde{U}_k$  which is not mapped into  $C(0, U_{k+1})$ . We also have

$$n_k = 2 + \sum_{j=0}^{k+1} q_j \quad \text{and} \quad m_k = 2 + \sum_{j=0}^k q_j$$

by (5.5). Thus (1.4) is satisfied. Note that  $n_k > m_k$ .

In order to prove (1.12) we note that by (5.4) the annulus

$$A_{k+1} = \text{ann}(4a_{k+1}, \sqrt{a_{k+2}})$$

separates  $f(c_k)$  and  $C(0, U_{k+1})$ . Now

$$\frac{\text{mod}(A_{k+1})}{n_k - m_k} = \frac{\log(\sqrt{a_{k+2}}/4a_{k+1})}{2\pi q_{k+1}} = \frac{q_{k+1}/2 - \log 4 - q_k}{2\pi q_{k+1}} \rightarrow \frac{1}{4\pi}$$

as  $k \rightarrow \infty$ , and this proves (1.12).

It remains to prove (5.1)–(5.4). In order to do this, we note first that  $q_{k+1} \geq 3q_0 q_k/2$  and thus  $q_k \geq (3q_0/2)^{k-j} q_j$  for  $k > j$ . Thus

$$\sum_{j=0}^{k-1} q_j \leq q_{k-1} \sum_{j=0}^{k-1} \left(\frac{2}{3q_0}\right)^j$$

which implies that given  $\varepsilon > 0$  we can achieve

$$(5.10) \quad \sum_{j=0}^{k-1} q_j \leq (1 + \varepsilon) q_{k-1}$$

by choosing  $q_0$  large. In particular,  $q_{k-2} \leq \varepsilon q_{k-1}$ .

We also note that the sequence  $(a_k)$  tends to  $\infty$  very rapidly and using this it is not difficult to see that we can achieve

$$(5.11) \quad \prod_{j=k+1}^{\infty} \left(1 + \frac{4a_k}{a_j}\right)^{q_j} \leq 2 \quad \text{and} \quad \prod_{j=k+1}^{\infty} \left(1 - \frac{4a_k}{a_j}\right)^{q_j} \geq \frac{1}{2}$$

for all  $k$  by choosing  $q_0$  large. We can also achieve

$$(5.12) \quad 1 + \frac{a_k}{4a_j} \leq a_k \quad \text{and} \quad \frac{a_k}{4a_j} - 1 \geq \frac{a_k}{5a_j}$$

for  $0 \leq j < k$ . For  $k \geq 1$  and  $|z| = a_k/4$  we then have

$$\begin{aligned} |f(z)| &\leq \frac{1}{16} a_k^2 \prod_{j=0}^{k-1} \left(1 + \frac{a_k}{4a_j}\right)^{q_j} \left(\frac{5}{4}\right)^{q_k} \prod_{j=k+1}^{\infty} \left(1 + \frac{a_k}{4a_j}\right)^{q_j} \\ &\leq \frac{1}{8} a_k^2 \prod_{j=0}^{k-1} a_k^{q_j} \left(\frac{5}{4}\right)^{q_k} \\ &= \frac{1}{8} \exp \left( 2q_{k-1} + q_{k-1} \sum_{j=0}^{k-1} q_j + q_k \log \frac{5}{4} \right) \\ &\leq \frac{1}{8} \exp \left( 2\varepsilon q_k + (1 + \varepsilon) q_{k-1}^2 + q_k \log \frac{5}{4} \right) \\ &= \frac{1}{8} \exp \left( \left( 2\varepsilon + (1 + \varepsilon) \frac{2}{3} + \log \frac{5}{4} \right) q_k \right) \end{aligned}$$

by (1.11), (5.10), (5.11) and (5.12). Choosing  $\varepsilon$  small, which we can achieve by choosing  $q_0$  large, we obtain

$$(5.13) \quad |f(z)| \leq \frac{1}{8} \exp q_k = \frac{1}{8} a_{k+1} \quad \text{for} \quad |z| = \frac{1}{4} a_k.$$

For  $k \geq 1$  and  $|z| = a_k/4$  we also have

$$\begin{aligned} |f(z)| &\geq \frac{1}{16} a_k^2 \prod_{j=0}^{k-1} \left(\frac{a_k}{4a_j} - 1\right)^{q_j} \left(\frac{3}{4}\right)^{q_k} \prod_{j=k+1}^{\infty} \left(1 - \frac{a_k}{4a_j}\right)^{q_j} \\ &\geq \frac{1}{32} a_k^2 \prod_{j=1}^{k-1} \left(\frac{a_k}{5a_j}\right)^{q_j} \left(\frac{3}{4}\right)^{q_k} \\ &= \frac{a_k}{32} \exp \left( q_{k-1} + \sum_{j=1}^{k-1} q_j (q_{k-1} - q_{j-1} - \log 5) + q_k \log \frac{3}{4} \right). \end{aligned}$$

by (5.11) and (5.12). As before we see that we also have

$$q_{k-1} + \sum_{j=1}^{k-1} q_j (q_{k-1} - q_{j-1} - \log 5) \geq q_k (1 - \varepsilon) \frac{2}{3}$$

for large  $q_0$  and since we may choose  $\varepsilon$  such that  $(1 - \varepsilon)2/3 + \log(3/4) > 0$ , we see that

$$(5.14) \quad |f(z)| \geq 4a_k \quad \text{for} \quad |z| = \frac{1}{4} a_k.$$

For  $k \geq 1$  and  $|z| = 4a_k$  we have

$$|f(z)| \geq 16a_k^2 \prod_{j=0}^{k-1} \left( \frac{4a_k}{a_j} - 1 \right)^{q_j} 3^{q_k} \prod_{j=k+1}^{\infty} \left( 1 - \frac{4a_k}{a_j} \right)^{q_j}$$

and thus

$$(5.15) \quad |f(z)| \geq 8 \cdot 3^{q_k} \geq 4a_{k+1} \quad \text{for } |z| = 4a_k$$

by (5.11). If  $|z| = 4a_0$ , then

$$|f(z)| \geq 16a_0^2 3^{q_0} \prod_{j=1}^{\infty} \left( 1 - \frac{4a_0}{a_j} \right)^{q_j} \geq 8a_0^2 3^{q_0} = 8a_1 3^{q_0} \geq 4a_1$$

so that (5.15) also holds for  $k = 0$ . By (5.13) we have  $|f(z)| \leq a_{k+2}/4$  for  $|z| = a_{k+1}/4$ . Since  $a_{k+1}/4 > 4a_k$  for large  $q_0$  we see that we also have

$$(5.16) \quad |f(z)| \leq \frac{1}{4}a_{k+2} \quad \text{for } |z| = 4a_k.$$

Now (5.1) follows from (5.13)–(5.16).

For  $k \geq 1$  and  $|z| = \sqrt{a_k}$  we see as before that

$$\begin{aligned} |f(z)|^2 &\leq a_k^2 \prod_{j=0}^{k-1} \left( 1 + \frac{\sqrt{a_k}}{a_j} \right)^{2q_j} \left( 1 + \frac{1}{\sqrt{a_k}} \right)^{2q_k} 2 \\ &\leq 2a_k^2 \prod_{j=0}^{k-1} a_k^{q_j} (1 + \varepsilon)^{2q_k} \\ &= \exp \left( \log 2 + 2q_{k-1} + q_{k-1} \sum_{j=0}^{k-1} q_j + 2q_k \log(1 + \varepsilon) \right) \\ &\leq \exp \left( \left( (1 + \varepsilon) \frac{2}{3} + 3\varepsilon + 2 \log(1 + \varepsilon) \right) q_k \right) \\ &\leq a_{k+1} \end{aligned}$$

for large  $q_0$  and small  $\varepsilon$ . This yields (5.2).

The location of the critical points could be determined by Rouché's theorem as in the proof of Theorem 1.2. Alternatively, to prove (5.3) it suffices that with

$$x_k = \frac{q_k}{2q_{k+1}} a_{k+1} \quad \text{and} \quad y_k = \frac{2q_k}{q_{k+1}} a_{k+1}$$

we have

$$f'(x_k) > 0 \quad \text{and} \quad f'(y_k) < 0.$$

Since  $f(x) \geq 0$  for all  $x \in \mathbb{R}$  it suffices to obtain these inequalities with  $f'$  replaced by  $f'/f$ . We have

$$\frac{f'(z)}{f(z)} = \frac{2}{z} + \sum_{j=0}^{\infty} \frac{q_j}{z - a_j}.$$

Thus

$$\frac{f'(x_k)}{f(x_k)} > \sum_{j=k}^{\infty} \frac{q_j}{x_k - a_j} = \frac{q_k}{x_k - a_k} - \frac{q_{k+1}}{a_{k+1} - x_k} - \sum_{j=k+2}^{\infty} \frac{q_j}{a_j - x_k}.$$



For large  $q_0$  we have

$$\frac{q_k}{x_k - a_k} = \frac{2q_k q_{k+1}}{q_k a_{k+1} - 2q_{k+1} a_k} \geq \frac{7}{4} \frac{q_{k+1}}{a_{k+1}}$$

while

$$\frac{q_{k+1}}{a_{k+1} - x_k} = \frac{2q_{k+1}^2}{2q_{k+1} a_{k+1} - q_k a_{k+1}} \leq \frac{5}{4} \frac{q_{k+1}}{a_{k+1}}$$

and

$$\sum_{j=k+2}^{\infty} \frac{q_j}{a_j - x_k} \leq 2 \sum_{j=k+2}^{\infty} \frac{q_j}{a_j} \leq \frac{1}{4} \frac{q_{k+1}}{a_{k+1}}.$$

The last four inequalities yield that  $f'(x_k) > 0$ . Similarly we find for large  $q_0$  that

$$\begin{aligned} \frac{f'(y_k)}{f(y_k)} &\leq \frac{2}{y_k} + \sum_{j=0}^k \frac{q_j}{y_k - a_j} - \frac{q_{k+1}}{a_{k+1} - y_k} \\ &= \frac{q_{k+1}}{q_k a_{k+1}} + \sum_{j=0}^k \frac{q_j q_{k+1}}{2q_k a_{k+1} - q_{k+1} a_j} - \frac{q_{k+1}^2}{q_{k+1} a_{k+1} - 2q_k a_{k+1}} \\ &\leq \frac{q_{k+1}}{q_k a_{k+1}} + \frac{5}{8} \sum_{j=0}^k \frac{q_j q_{k+1}}{q_k a_{k+1}} - \frac{7}{8} \frac{q_{k+1}}{a_{k+1}} \\ &= \frac{q_{k+1}}{a_{k+1}} \left( \frac{1}{q_k} + \frac{5}{8} \frac{1}{q_k} \sum_{j=0}^k q_j - \frac{7}{8} \right) \\ &< 0 \end{aligned}$$

and thus  $f'(y_k) < 0$ . This completes the proof of (5.3).

The right inequality of (5.4) follows from (5.1) and (5.3). To prove the left one we note that

$$\begin{aligned} (5.17) \quad f(c_k) &\geq f(x_k) \\ &= x_k^2 \prod_{j=0}^k \left( \frac{x_k - 1}{a_j} \right)^{q_j} \left( 1 - \frac{x_k}{a_{k+1}} \right)^{q_{k+1}} \prod_{j=k+2}^{\infty} \left( 1 - \frac{x_k}{a_j} \right)^{q_j} \\ &\geq x_k^2 \left( \frac{x_k - 1}{a_k} \right)^{q_k} \left( 1 - \frac{x_k}{a_{k+1}} \right)^{q_{k+1}} \frac{1}{2} \\ &\geq \left( \frac{x_k}{2a_k} \right)^{q_k} \left( 1 - \frac{x_k}{a_{k+1}} \right)^{q_{k+1}} \\ &= \left( \frac{q_k a_{k+1}}{4q_{k+1} a_k} \right)^{q_k} \left( 1 - \frac{q_k}{2q_{k+1}} \right)^{q_{k+1}} \\ &\geq \left( \frac{q_k a_{k+1}}{4q_{k+1} a_k} \right)^{q_k} \exp(-q_{k-1}) \end{aligned}$$

for large  $q_0$ . Noting that

$$q_k = \frac{3}{2} q_{k-1}^2 = \frac{3}{2} (\log a_k)^2$$

and

$$a_k = \exp q_{k-1} = \exp \sqrt{\frac{2}{3} q_k} \leq \exp \sqrt{q_k} = \exp \sqrt{\log a_{k+1}}$$

we see that for given  $\delta > 0$  we can achieve

$$q_k \leq a_k^\delta \quad \text{and} \quad a_k \leq a_{k+1}^\delta$$

by choosing  $q_0$  large. Thus, given  $\varepsilon > 0$ , we deduce from (5.17) that

$$f(c_k) \geq a_{k+1}^{(1-\varepsilon)q_k} \exp(-q_k) = \exp((1-\varepsilon)q_k^2 - q_k) \geq \exp\left((1-2\varepsilon)\frac{2}{3}q_{k+1}\right)$$

for large  $q_0$ . For small  $\varepsilon > 0$  we thus have

$$f(c_k) \geq \exp\left(\frac{1}{2}q_{k+1}\right) = \sqrt{a_{k+2}}.$$

This completes the proof of (5.4) and thus the proof of Theorem 1.3.

**Remark 5.1.** *Let  $f$  be a transcendental entire function with a multiply connected wandering domain  $U_0$  and put  $U_k = f^k(U_0)$  as before. By Theorem A, part (iii), the  $U_k$  are all bounded and thus  $\partial U_k \cap C(\infty, U_k)$  is connected. We call  $\partial U_k \cap C(\infty, U_k)$  the outer boundary of  $U_k$  and denote it by  $\partial_\infty U_k$ . By Theorem A, part (ii), we have  $0 \in C(0, U_k)$  for large  $k$ , and for such  $k$  we call  $\partial U_k \cap C(0, U_k)$  the inner boundary of  $U_k$  and denote it by  $\partial_0 U_k$ . It is not difficult to see when  $U_k$  is infinitely connected, then the components of  $\partial U_k$  must cluster at the inner or outer boundary (or both). However, as we will explain below, it can happen that they cluster at only one of them.*

*In the proof of Theorem 1.3 we have shown that, with the terminology used there,  $X_k = D(0, \sqrt{a_{k+1}}) \setminus C(0, U_k) \subseteq U_k$ . This implies that the components of  $\partial U_k$  do not cluster at the inner boundary  $\partial_0 U_k$ : it is an isolated part of the boundary in the sense that  $\text{dist}(\partial_0 U_k, \partial U_k \setminus \partial_0 U_k) > 0$ .*

*On the other hand, if  $f$  is as in Theorem 1.2, with  $(P_k)$  chosen such that*

$$(5.18) \quad \limsup_{k \rightarrow \infty} kP_k < \frac{|C|}{2e},$$

*then the outer boundary of  $U_k$  is an isolated part of the boundary. In fact, the proof of Theorem 1.2 shows that (5.18) implies that (4.12) holds for all large  $k$ . We deduce that  $S_{k+1} = \widetilde{U_{k+1}} \setminus D(0, 2^k r_{k+1})$  contains no critical values. Thus the components of  $f^{-1}(S_{k+1})$  are doubly connected. Proceeding as in the proof of (4.5), we can show that  $|f(z)| \geq 2^k r_{k+1}$  for  $|z| = 2^{k-1} r_k$ . This implies that there exists a component  $T_k$  of  $f^{-1}(S_{k+1})$  which contains  $S_k$ . It follows that  $f(S_k) \subseteq f(T_k) = S_{k+1}$ , and this implies that  $S_k \subseteq F(f)$ . Thus the outer boundary  $\partial_\infty U_k$  of  $U_k$  is isolated.*

*A similar reasoning shows that if*

$$\liminf_{k \rightarrow \infty} kP_k > \frac{|C|}{2e},$$

*then  $\partial_0 U_k$  is isolated, and if*

$$\liminf_{k \rightarrow \infty} kP_k < \frac{|C|}{2e} < \limsup_{k \rightarrow \infty} kP_k,$$

*then neither  $\partial_0 U_k$  nor  $\partial_\infty U_k$  is isolated.*

*These examples illustrate a result in [9, Theorem 8.1] where it is shown that the inner or outer boundary is isolated under a suitable hypothesis on the location of the critical points.*

## 6. BAKER'S EXAMPLE OF AN INFINITELY CONNECTED FATOU COMPONENT

In Baker's example (1.3) the constant  $C$  and the sequence  $(r_k)$  are chosen as follows. First let  $0 < C < 1/(4e^2)$  and  $r_1 > 1$ . Then choose  $k_0$  such that  $2^{k_0-1}C > 2r_1$ . Finally choose  $(r_k)$  such that  $r_{k+1} \geq 2r_k$  for  $1 \leq k < k_0$  and put

$$r_{k+1} = C^2 \prod_{j=1}^k \left(1 + \frac{r_k}{r_j}\right)^2$$

for  $k \geq k_0$ .

Baker showed that with

$$s_k = \frac{k+1}{k+2}r_{k+1} \quad \text{and} \quad B_k = \text{ann}(r_k^2, s_k)$$

we have  $f(B_k) \subseteq B_{k+1}$  for large  $k$  and thus  $B_k \subseteq U_k$  for some multiply connected wandering domain  $U_k$ . In order to prove that  $U_k$  is infinitely connected, Baker proved that there exists a critical point

$$c_{k+1} \in (-s_k, -r_k^2) \subset B_k$$

for large  $k$ . The infinite connectivity then follows from Lemma 2.2.

Analogously to (4.9) we now find that

$$c_k = -\left(1 - \frac{1}{k + \delta_k}\right)r_k,$$

where  $\delta_k \rightarrow 0$ , and instead of (4.11) we now obtain

$$|f(c_k)| \sim \frac{C^2}{ek} \prod_{j=1}^{k-1} \left(1 + \frac{r_k}{r_j}\right)^2 = \frac{1}{4ek}r_{k+1}.$$

Similar to (4.1) we also have  $r_{k+1} \geq 2^k r_k^2$ . Hence  $|f(c_k)|/r_k^2 \rightarrow \infty$ . As  $c_{k+1}$  is the only critical point in  $U_k$ , it now follows from Theorem 1.1, applied with  $A_{k+1} = \text{ann}(r_{k+1}^2, |f(c_{k+1})|) \subseteq U_{k+1}$ , that  $\partial U_k$  is not uniformly perfect.

## REFERENCES

- [1] I. N. Baker, Multiply connected domains of normality in iteration theory. *Math. Z.* 81 (1963), 206–214.
- [2] I. N. Baker, The domains of normality of an entire function. *Ann. Acad. Sci. Fenn. Ser. A I Math.* 1 (1975), 277–283.
- [3] I. N. Baker, An entire function which has wandering domains. *J. Australian Math. Soc. (Ser. A)* 22 (1976), 173–176.
- [4] I. N. Baker, Wandering domains in the iteration of entire functions. *Proc. London Math. Soc.* (3) 49 (1984), 563–576.
- [5] I. N. Baker, Infinite limits in the iteration of entire functions. *Ergodic Theory Dynamical Systems* 8 (1988), 503–507.
- [6] A. F. Beardon and Ch. Pommerenke, The Poincaré metric of plane domains. *J. London Math. Soc.* (2) 18 (1978), 475–483.
- [7] W. Bergweiler, Iteration of meromorphic functions. *Bull. Amer. Math. Soc. (N. S.)* 29 (1993), 151–188.
- [8] W. Bergweiler, Connectivity of Fatou components. In “Mini-Workshop: The escaping set in transcendental dynamics”, Oberwolfach Rep. 6 (2009), 2927–2963.
- [9] W. Bergweiler, P. J. Rippon and G. M. Stallard, Multiply connected wandering domains of entire functions. Preprint, arXiv: 1109.1794.
- [10] A. Hinkkanen, Julia sets of rational functions are uniformly perfect, *Math. Proc. Cambridge Philos. Soc.* 113 (1993), 543–559.

- [11] M. Kisaka and M. Shishikura, On multiply connected wandering domains of entire functions. In “Transcendental dynamics and complex analysis”, edited by P. J. Rippon and G. M. Stallard, LMS Lecture Note Series 348, Cambridge University Press, 2008, 217–250.
- [12] R. Mañé and L. F. da Rocha, Julia sets are uniformly perfect. Proc. Amer. Math. Soc. 116 (1992), 251–257.
- [13] C. T. McMullen, Complex dynamics and renormalization. Ann. of Math. Studies 135, Princeton Univ. Press, Princeton, NJ, 1994.
- [14] Ch. Pommerenke, Uniformly perfect sets and the Poincaré metric. Arch. Math. (Basel) 32 (1979), 192–199.
- [15] D. Schleicher, Dynamics of entire functions. In “Holomorphic dynamical systems”, Lecture Notes Math. 1998 (2010), 295–339.
- [16] N. Steinmetz, Rational iteration. Walter de Gruyter, Berlin, 1993.
- [17] T. Sugawa, Various domain constants related to uniform perfectness. Complex Variables Theory Appl. 36 (1998), 311–345.
- [18] D. Sullivan, Quasiconformal homeomorphisms and dynamics I. Solution of the Fatou-Julia problem on wandering domains. Ann. of Math. 122 (1985), 401–418.
- [19] H. Töpfer, Über die Iteration der ganzen transzendenten Funktionen, insbesondere von  $\sin z$  und  $\cos z$ . Math. Ann. 117 (1939), 65–84.
- [20] J. H. Zheng, On uniformly perfect boundaries of stable domains in iteration of meromorphic functions. Bull. London Math. Soc. 32 (2000), 439–446.
- [21] J. H. Zheng, Uniformly perfect sets and distortion of holomorphic functions. Nagoya Math. J. 164 (2001), 17–33.
- [22] J. H. Zheng, On multiply-connected Fatou components in iteration of meromorphic functions. J. Math. Anal. Appl. 313 (2006), 24–37.

MATHEMATISCHES SEMINAR, CHRISTIAN-ALBRECHTS-UNIVERSITÄT, ZU KIEL, LUDEWIG-MEYN-STR. 4, D-24098 KIEL, GERMANY

*E-mail address:* `bergweiler@math.uni-kiel.de`

DEPARTMENT OF MATHEMATICAL SCIENCES, TSINGHUA UNIVERSITY, 100084, BEIJING, P. R. CHINA

*E-mail address:* `jzheng@math.tsinghua.edu.cn`