ITERATION OF QUASIREGULAR MAPPINGS

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Abstract. We survey some results on the iteration of quasiregular mappings. In particular we discuss some recent results on the dynamics of quasiregular maps which are not uniformly quasiregular.

1. Introduction

Complex dynamics is concerned with the behavior of meromorphic functions under iteration. The theory was initiated by Fatou [28] and Julia [41], who wrote long memoirs on the iteration of rational functions between 1918 and 1920. Fatou [29] extended some of the results to entire functions in 1926. After half a century of comparatively little activity, complex dynamics became an area of intensive research in the 1980s, when Sullivan [75, 76], Douady and Hubbard [24, 25], and others introduced powerful new mathematical techniques to the subject, most notably quasiconformal mappings. Moreover, computer graphics of Julia sets, the Mandelbrot set and other sets related to iteration created interest in the subject even among non-mathematicans.

More recently, there has been interest in whether the concepts and ideas of complex dynamics can be extended to certain classes of non-holomorphic functions, and here in particular to quasiregular maps, both in dimension 2 and in higher dimensions. This paper describes some of the results that have been obtained in this context.

We begin with a short (and very incomplete) overview of complex dynamics in section 2, restricting ourselves primarily to results which will be relevant in subsequent sections. Then we introduce and discuss quasiregular maps in section 3. The dynamics of quasiregular maps are discussed in two subsequent sections, distinguishing the cases whether the map is uniformly quasiregular (section 4) or not (section 5).

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2. Complex dynamics

2.1. **Rational functions.** The basic objects studied in the iteration theory of rational functions developed by Fatou [28] and Julia [41] are two sets today named after them, namely the *Fatou set*

$$F(f) := \left\{ z \in \overline{\mathbb{C}} : \left\{ f^k \right\} \text{ is normal at } z \right\}$$

and the *Julia set*

$$J(f) := \overline{\mathbb{C}} \setminus F(f).$$

Here $\overline{\mathbb{C}} := \mathbb{C} \cup \{\infty\}$ denotes the Riemann sphere and f^k is the k-th iterate of the rational function f. One assumes throughout that f is neither constant nor linear.

The Fatou-Julia theory was one of the first applications of the concept of normality introduced by Montel [57]. In fact, Montel's theorem that a family of meromorphic functions omitting three fixed values is normal plays a major role in complex dynamics.

Before we list some basic properties of the Fatou and Julia sets, we introduce some further notation. For $z \in \overline{\mathbb{C}}$ we call

$$O^+(z) := \{ f^n(z) : n \geq 0 \}$$

the *forward orbit* of z,

$$O^-(z) := \bigcup_{n \geq 0} f^{-n}(z) = \bigcup_{n \geq 0} \left\{ w \in \overline{\mathbb{C}} : f^n(w) = z \right\}$$

the *backward orbit* of z and

$$O(z) := O^+(z) \cup O^-(z)$$

the *orbit* of z. For $A \subset \overline{\mathbb{C}}$ we put $O^{(\pm)}(A) := \bigcup_{z \in A} O^{(\pm)}(z)$. We say that A is *completely invariant* if $O(A) = A$. The *exceptional set* $E(f)$ is defined as the set of all points whose backward orbit is finite.

We say that $\xi \in \overline{\mathbb{C}}$ is *periodic* if there exists $p \in \mathbb{N}$ such that $f^p(\xi) = \xi$. The smallest p with this property is called the *period* of ξ. For a periodic point $\xi \neq \infty$ of period p we call $(f^p)'(\xi)$ the *multiplier* of ξ. If $\xi = \infty$, the multiplier is given by $\frac{d}{dz} \left(1/f(1/z) \right) |_{z=0}$. A periodic point is called *attracting*, *indifferent* or *repelling* depending on whether the modulus of its multiplier is less than, equal to or greater than 1. If ξ is an attracting periodic point of period p, then

$$A(\xi) := \left\{ w \in \overline{\mathbb{C}} : \lim_{n \to \infty} f^{pn}(w) = \xi \right\}$$

is called the *basin of attraction* of ξ. A periodic point of period one is called a *fixed point*. Finally, we denote by $|X|$ the cardinality of a set X and we recall that a subset of a metric space is called *perfect* if it is non-empty, closed and does not contain isolated points.

The following theorem summarizes some of the results obtained by Fatou and Julia.

Theorem 2.1. *Let f be a rational function of degree at least 2. Then*

(i) *$F(f)$ is open and $J(f)$ is closed;*
(ii) *$F(f^n) = F(f)$ and $J(f^n) = J(f)$ for all $n \in \mathbb{N}$;*
(iii) *$F(f)$ and $J(f)$ are completely invariant;*
(iv) *$J(f)$ is perfect;*
(v) *if $X \subset \overline{\mathbb{C}}$ is closed and completely invariant and $|X| \geq 3$, then $X \supset J(f)$;*
(vi) *if ξ is an attracting periodic point, then $A(\xi) \subset F(f)$ and $\partial A(\xi) = J(f)$;*
(vii) *if U is open and $U \cap J(f) \neq \emptyset$, then $O^+(U) \supset \overline{\mathbb{C}} \setminus E(f)$;*
(viii) *$|E(f)| \leq 2$ and $E(f) \cap J(f) = \emptyset$;*
(ix) *if $z \in J(f)$ then $J(f) = \overline{O^-(z)}$;*

The proofs of the first three properties are fairly elementary. Property (iv) implies in particular that the Julia set is not empty. Once this is known, one way to prove that a given point $\xi \in J(f)$ is not isolated in $J(f)$ consists of showing that there exists a point in $J(f) \setminus O^+(\xi)$ and applying (ix) to this point. Properties (v)–(vii) and (ix) are all applications of Montel's theorem.

We comment on some of these properties. Property (ix) leads in obvious way to an algorithm to produce computer pictures of the Julia set, at least when the inverse function can be computed, for example for polynomials of low degree. Property (vi) is one indication why Julia sets are often so complicated: if f has three distinct attracting periodic points ξ_1, ξ_2 and ξ_3, then the corresponding attracting basins are disjoint but have a common boundary. Finally we note that it follows from property (vii) – together with (iii) and (viii) – that if U is an open set intersecting $J(f)$, then $O^+(U \cap J(f)) = J(f)$. This explains why Julia sets are "self-similar".

Another basic result of the theory is the following result.

Theorem 2.2. *The Julia set of a rational function is the closure of the set of repelling periodic points.*

One reason why Theorem 2.2 is of interest is historical. While Fatou started his investigations by considering the set of non-normality – the approach that we have taken above – Julia built his theory on the closure of the set of repelling periodic points. Both of them eventually proved that these two sets are equal (see [28, Section 30, p. 69] and [41, p. 99, p. 118]), but their approaches to this result and thus their proofs of Theorem 2.2 were completely different. A good exposition of both proofs can be found in [55, Section 11].

We note, however, that Theorem 2.2 is of interest not only from the historical point of view, it also has some important consequences. One corollary is that if U is an open set intersecting the Julia set, then no subsequence of (f^n) is normal. Another consequence of Theorem 2.2 is a sharpening of the above result that $O^+(U \cap J(f)) = J(f)$ if U is an open set intersecting $J(f)$. Noting that U contains a repelling periodic point, say of period p, and that for a small neighborhood V of this periodic point we have $\overline{V} \subset f^p(V)$, we obtain the following result.

Theorem 2.3. *Let f be rational of degree at least 2. If U is an open set intersecting $J(f)$, then $f^n(U \cap J(f)) = J(f)$ for all large n.*

If U_0 is a component of $F(f)$ and $n \in \mathbb{N}$, then $U_n := f^n(U_0)$ is also a component of $F(f)$. We say that U_0 is *periodic* if $U_0 = U_p$ for some $p \in \mathbb{N}$ and call the smallest p with this property the period of U_0. We call U_0 *preperiodic* if there exists $q \in \mathbb{N}$ such that U_q is periodic. A component of period 1 is called *invariant*. A famous theorem of Sullivan [74, 76] says that all components of $F(f)$ are preperiodic for a rational function f. It was in the proof of this result where quasiconformal mappings were first introduced to complex dynamics, the main tool being Theorem 3.3 below; cf. [25, p. 294] and [76, p. 406].

In order to understand the dynamics on the Fatou set it thus suffices to consider periodic components. Using property (ii) of Theorem 2.1 one may in fact restrict to invariant components. The dynamics in such components are completely described by the following result.

Theorem 2.4. *Let U be an invariant component of $F(f)$. Then we have one of the following possibilities:*

- *U contains an attracting fixed point ξ and $f^n|_U \to \xi$ as $n \to \infty$,*
- *∂U contains a fixed point ξ of multiplier 1 and $f^n|_U \to \xi$ as $n \to \infty$,*
- *$f : U \to U$ is bijective and there exists a increasing sequence (n_k) such that $f^{n_k}|_U \to id_U$ as $k \to \infty$.*

In this form the theorem is due to Fatou [28, Section 56, p. 249]. Cremer [20, p. 317] proved that in the last case $f|_U$ is actually conjugate to an irrational rotation of a disk or an annulus. Therefore such domains are called *rotation domains*. Also, in the case of the disk the domain U is called a *Siegel disk* and in the case of the annulus it is called a

Herman ring. We note that when Julia, Fatou and Cremer proved their results, it was not known whether rotation domains actually exist.

All the above results – and much more – can be found in the books by Beardon [8], Carleson and Gamelin [18], Milnor [55] and Steinmetz [73] as well as the survey by Eremenko and Lyubich [27], to all of which we refer for a thorough introduction to the iteration theory of rational functions.

2.2. **Entire functions.** The iteration of transcendental entire functions was considered first by Fatou [29] in 1926. In part the theory runs parallel to that for rational functions, but there are also some significant differences; see [10], [27, Chapter 4] and [58, Chapter 3] for a more detailed treatment of the iteration theory of entire functions.

The definition of the Fatou set, the Julia set and the exceptional set are the same as before, except that they are considered as subsets of the complex plane and not the Riemann sphere now.

With these definitions it turns out that properties (i)–(iv) and (vi) of Theorem 2.1 remain true literally, although some proofs need modification. In particular, Fatou's proof that $J(f) \neq \emptyset$ was much more involved than in the case of rational functions. A simpler proof – which, however, is still more complicated than in the rational case – has been given by Bargmann [6].

Properties (v), (vii), (viii) and (ix) have to be modified slightly. In (v) we only consider subsets X of $\mathbb{C}$, but it suffices to assume that $|X| \geq 2$. In (vii) we obtain $O^+(U) \supset \mathbb{C} \setminus E(f)$. In (viii) we have $|E(f)| \leq 1$, but it is possible that $E(f) \cap J(f) \neq \emptyset$. In (ix) we can now conclude that $J(f) = \overline{O^-(z)}$ only for $z \in J(f) \setminus E(f)$. As an example we consider $f(z) := \lambda z e^z$ with $|\lambda| > 1$. Then 0 is a repelling fixed point and thus in $J(f)$. Clearly we also have $0 \in E(f)$.

As mentioned, Fatou and Julia had proved Theorem 2.2 for rational functions by different methods. However, neither proof generalizes to the case of transcendental entire functions. It was only in 1968 when Baker [4] succeeded in proving Theorem 2.2 for transcendental entire functions. His proof was based on the Ahlfors theory of covering surfaces, in particular the Ahlfors five islands theorem; see [1], [32, Chapter 5] or [59, Chapter XIII]. A more elementary proof based on Zalcman's lemma [84, 85] was later given by Schwick [70]. This proof has been further simplified by Bargmann [6] and Berteloot and Duval [17]. We mention, however, that meanwhile a fairly elementary proof of the Ahlfors five islands theorem based on Zalcman's lemma is also available [11]. Ahlfors's theorem actually has a number of further applications in complex dynamics [12].

An important characterization of the Julia set of entire functions was given by Eremenko [26] who undertook a thorough study of the *escaping set*

$$I(f) := \{z \in \mathbb{C} : \lim_{n\to\infty} |f^n(z)| = \infty\}.$$

For polynomials the point at ∞ is an attracting fixed point and thus $I(f) \subset F(f)$ and $J(f) = \partial I(f)$ by property (vi) of Theorem 2.1. Eremenko showed that $J(f) = \partial I(f)$ also holds for transcendental entire functions. The main step is to prove that $I(f) \neq \emptyset$. Once this is known, it follows fairly easily that $J(f) = \partial I(f)$. Eremenko's proof that $I(f) \neq \emptyset$ was based on Wiman-Valiron theory (see [33] and also [16]). An alternative proof was given by Domínguez [23].

Eremenko also proved that if f is transcendental entire, then $I(f) \cap J(f) \neq \emptyset$ and $\overline{I(f)}$ does not have bounded components. He asked whether

(a) every component of $I(f)$ is unbounded

or – even stronger – whether

(b) every point in $I(f)$ can be connected to ∞ by a curve in $I(f)$.

These questions have prompted a considerable amount of research. It was recently shown by Rottenfußer, Rückert, Rempe and Schleicher [66, Theorem 1.1] that the answer to (b) is negative: there exists a transcendental entire function f for which every path-connected component of $I(f)$ is a point. On the other hand, it was shown by the same authors [66, Theorem 1.2] and, independently, by Barański [5] that the answer to (b) – and hence to (a) – is positive for a large class of functions. Question (a) is still open, but Rippon and Stallard [65] have shown that $I(f)$ contains *at least one* unbounded component.

Eremenko's questions were motivated by examples considered by various authors, in particular by results of Devaney and Krych [21] dealing with the functions $E_\lambda(z) := \lambda e^z$. They had shown that for $0 < \lambda < 1/e$ the Julia set of E_λ is a "Cantor set of curves". To give a formal statement of their result we say that a subset H of $\mathbb{C}$ is a *(Devaney) hair* if there exists a homeomorphism $\gamma : [0,\infty) \to H$ such that $\gamma(t) \to \infty$ as $t \to \infty$. We call $\gamma(0)$ the *endpoint* of the hair H. Devaney and Krych [21] proved that if $0 < \lambda < 1/e$, then $J(E_\lambda)$ is an uncountable union of pairwise disjoint hairs. Moreover, if C_λ denotes the set of endpoints of the hairs that form $J(E_\lambda)$, then $J(E_\lambda) \setminus C_\lambda \subset I(f)$. Thus (b) clearly holds for such maps. Schleicher and Zimmer [69] proved that (b) holds if $f = E_\lambda$ for all $\lambda \in \mathbb{C} \setminus \{0\}$.

We also mention two results concerning the dimension of these sets. Here and in the following we denote the Hausdorff dimension of a subset

A of $\mathbb{C}$ (or of $\mathbb{R}^d$) by $\dim A$. McMullen [54, Theorem 1.2] proved that $\dim J(E_\lambda) = 2$ for all $\lambda \in \overline{\mathbb{C}}\backslash\{0\}$ while Karpińska [43, Theorem 1.1] obtained the seemingly paradoxical results that $\dim(J(E_\lambda)\backslash C_\lambda) = 1$ if $0 < \lambda < 1/e$.

Similar results have been obtained, largely by the same authors, also for the maps $S_{\alpha,\beta}(z) := \alpha \sin z + \beta$, where $\alpha, \beta \in \mathbb{C}$, $\alpha \neq 0$. McMullen [54, Theorem 1.1] showed that $J(S_{\alpha,\beta})$ has positive measure, Devaney and Tangerman [22] showed that $J(S_{\alpha,0})$ consists of hairs for $0 < \alpha < 1$ and Karpińska [42, Theorem 3] showed that the set of endpoints of these hairs has positive measure. If α and β are chosen such that the critical values of f are preperiodic, then $J(S_{\alpha,\beta}) = \mathbb{C}$. Schleicher [68] showed that hairs can still be defined and that the set of hairs without endpoints has Hausdorff dimension 1. In particular it follows that the set of endpoints has full measure. Thus he obtains a representation of the plane as a union of hairs such that the intersection of any two hairs is either empty or consists of the common endpoint and the union of the hairs without their endpoints has Hausdorff dimension 1.

3. Quasiregular maps

3.1. **Quasiregular maps in space.** We introduce quasiregular maps only briefly and refer to Rickman's monograph [64] for a detailed treatment.

Let $d \geq 2$ be an integer and let $\Omega \subset \mathbb{R}^d$ be a domain. By $ACL(\Omega)$ we denote the set of all continuous maps $f = (f_1, \ldots, f_d) : \Omega \to \mathbb{R}^d$ which are absolutely continuous on almost all lines parallel to the coordinate axes. For $f \in ACL(\Omega)$ the partial derivatives $\partial_k f_j$ exist almost everywhere. For $p \geq 1$ we denote by $ACL^p(\Omega)$ the set of all $f \in ACL(\Omega)$ for which all partial derivatives are locally L^p-integrable. The definition of $ACL^p(\Omega)$ does not depend on the choice of coordinates [64, Proposition I.1.11].

It turns out [64, Proposition I.1.4] that a continuous map $f : \Omega \to \mathbb{R}^d$ is in $ACL^p(\Omega)$ if and only if it belongs to the Sobolev space $W^1_{p,loc}(\Omega)$. By definition, this space consists of all functions $f : \Omega \to \mathbb{R}^d$ for which all first order weak partial derivatives $\partial_k f_j$ exist and are locally in L^p.

Denote the (euclidean) norm of $x \in \mathbb{R}^d$ by $|x|$. A map $f \in ACL^d(\Omega)$ is called *quasiregular* if there exists a constant $K_O \geq 1$ such that

$$|Df(x)|^d \leq K_O\, J_f(x) \quad \text{a.e.,} \tag{3.1}$$

where $Df(x)$ denotes the derivative,

$$|Df(x)| := \sup_{|h|=1} |Df(x)(h)|$$

its norm, and $J_f(x)$ the Jacobian determinant. With

$$\ell(Df(x)) := \inf_{|h|=1} |Df(x)(h)|$$

the condition that (3.1) holds for some $K_O \geq 1$ is equivalent to the condition that

$$J_f(x) \leq K_I \, \ell(Df(x))^d \quad \text{a.e.,} \tag{3.2}$$

for some $K_I \geq 1$. The smallest constants $K_O = K_O(f)$ and $K_I = K_I(f)$ for which (3.1) and (3.2) hold are called the *outer and inner dilatation* of f. Moreover, $K(f) := \max\{K_I(f), K_O(f)\}$ is called the (maximal) *dilatation* of f. We say that f is K-*quasiregular* if $K(f) \leq K$. An injective K-quasiregular map is called K-*quasiconformal* (or just *quasiconformal*) and a 1-quasiconformal map is called *conformal*. It turns out (see [61, Theorem 5.10] or [64, Theorem I.2.5]) that for $d \geq 3$ the only conformal maps are restrictions of Möbius transformations. These are – by definition – compositions of an even number of reflections at hyperplanes or spheres.

The *branch set* B_f of a quasiregular map f consists of all points where f is not locally injective. We note that a theorem of Zorich [86] says that if $d \geq 3$ and $f : \mathbb{R}^d \to \mathbb{R}^d$ is quasiregular and not injective, then $B_f \neq 0$. In fact, in this case the $(d-2)$-dimensional Hausdorff measure of $f(B_f)$ is positive [64, Theorem III.5.3].

Next we note that if f and g are quasiregular, then the composition $f \circ g$ is also quasiregular, assuming of course that the domain of f contains the range of g. More precisely, we have $K(f \circ g) \leq K(f)K(g)$.

Quasiregularity can be defined more generally for maps between Riemannian manifolds. Here we only consider the case that the domain or range are equal to (or contained in) the one point compactification $\overline{\mathbb{R}^d} := \mathbb{R}^d \cup \{\infty\}$ of $\mathbb{R}^d$, endowed with the spherical metric obtained via stereographical projection from the unit sphere in $\mathbb{R}^{d+1}$. It turns out that for $\Omega \subset \overline{\mathbb{R}^d}$ a non-constant continuous map $f : \Omega \to \overline{\mathbb{R}^d}$ is quasiregular if $f^{-1}(\infty)$ is discrete and if f is quasiregular in $\Omega \setminus (f^{-1}(\infty) \cup \{\infty\})$.

Many properties of holomorphic maps hold for quasiregular maps as well. For example, non-constant quasiregular maps are open and discrete. We do not survey here the many other results of complex function theory that have analogues for quasiregular maps, but only mention the analogue of Picard's theorem which is due to Rickman [62].

Theorem 3.1. *Let $d \in \mathbb{N}$, $d \geq 2$, and $K \geq 1$. There exists $q = q(d, K)$ with the following property: if $a_1, \dots, a_q \in \mathbb{R}^d$ are distinct and if $f : \mathbb{R}^d \to \mathbb{R}^d \setminus \{a_1, \dots, a_q\}$ is K-quasiregular, then f is constant.*

Equivalently, we can say that a non-constant K-quasiregular map $f : \mathbb{R}^d \to \overline{\mathbb{R}^d}$ omits at most q values. Note that Picard's theorem says that $q(2, 1) = 2$.

Miniowitz [56] has used an extension of the Zalcman lemma [84] to quasiregular maps to obtain an analogue of Montel's theorem from Rickman's result.

Theorem 3.2. *Let $d \in \mathbb{N}$, $d \geq 2$, and $K \geq 1$. Let $a_1, \dots, a_q \in \mathbb{R}^d$ be distinct, where $q = q(d, K)$ is as in Rickman's theorem. Let $\Omega \subset \mathbb{R}^d$ be a domain. Then the family of all K-quasiregular maps $f : \Omega \to \mathbb{R}^d \setminus \{a_1, \dots, a_q\}$ is normal.*

Equivalently, if $a_1, \dots, a_{q+1} \in \overline{\mathbb{R}^d}$ are distinct, then the family of all K-quasiregular maps $f : \Omega \to \overline{\mathbb{R}^d} \setminus \{a_1, \dots, a_{q+1}\}$ is normal. Similarly, we may allow that $\Omega \subset \overline{\mathbb{R}^d}$.

We consider two examples of quasiregular maps. We begin with the so-called *winding map*; cf. [64, Section I.3.1].

Example 3.1. Let $N \in \mathbb{N}$, $N \geq 2$, and let $f : \mathbb{R}^3 \to \mathbb{R}^3$ be the map which in cylindrical coordinates is defined by $(r, \theta, x_3) \mapsto (r, N\theta, x_3)$.

A computation shows that if $x_3 \neq 0$, then f is differentiable at $x = (x_1, x_2, x_3)$ and we have $|Df(x)| = N$, $\ell(Df(x)) = 1$ and $J_f(x) = N$. Thus f is quasiregular with $K_I(f) = N$ and $K_O(f) = N^2$. We note that the branch set B_f is the x_3-axis.

Another important example of a quasiregular map $Z : \mathbb{R}^3 \to \mathbb{R}^3$ was given by Zorich [86, p. 400]; see also [40, Section 6.5.4] and [64, Section I.3.3]. This map can be considered as a three-dimensional analogue of the exponential function. In our description of Zorich's map we follow [40].

Example 3.2. Consider the square

$$Q := \left\{(x_1, x_2) \in \mathbb{R}^2 : \ |x_1| \leq 1, |x_2| \leq 1\right\}$$

and the upper hemisphere

$$U := \left\{(x_1, x_2, x_3) \in \mathbb{R}^3 : \ x_1^2 + x_2^2 + x_3^2 = 1, \ x_3 \geq 0\right\}.$$

Let $h : Q \to U$ be a bilipschitz map and define

$$Z : Q \times \mathbb{R} \to \mathbb{R}^3, \ Z(x_1, x_2, x_3) = e^{x_3} h(x_1, x_2).$$

Then Z maps the "infinite square beam" $Q \times \mathbb{R}$ to the upper half-space. Repeated reflection along sides of square beams and the (x_1, x_2)-plane now yields a map $Z : \mathbb{R}^3 \to \mathbb{R}^3$. It turns out that this map Z is quasiregular. Its dilatation is bounded in terms of the bilipschitz constant of h. We will call a map Z obtained this way a *Zorich map*.

Note that $Z(x_1 + 4, x_2, x_3) = Z(x_1, x_2 + 4, x_3) = Z(x_1, x_2, x_3)$ for all $x \in \mathbb{R}^3$ so that Z is "doubly periodic". We also mention that the branch set of Z is given by the edges of the square beam and the lines obtained from them by reflections; that is, $B_Z = \{(2m+1, 2n+1, x_3) : m, n \in \mathbb{Z}, x_3 \in \mathbb{R}\}$. Moreover, $Z(x) \neq 0$ for all $x \in \mathbb{R}^3$.

Since Zorich's map omits 0 we see that the constant from Rickman's theorem satisfies $q(3, K) \geq 2$ for large K. However, we mention that Rickman [63] has actually shown that $q(3, K) \to \infty$ as $K \to \infty$.

We have restricted above to dimension 3 only for simplicity. Both the winding map and the Zorich map can also be defined in dimension greater than 3; see [64, Section I.3.1] for the winding map and [50] for the Zorich map.

3.2. **Quasiregular maps in the plane.** The theory of quasiregular and quasiconformal maps is much more advanced in dimension 2; see [3, 45] for a thorough treatment of quasiconformal mappings in the plane. Using the Wirtinger notation f_z and $f_{\overline{z}}$ we have

$$|Df(z)| = |f_z(z)| + |f_{\overline{z}}(z)|,$$
$$\ell(Df(z)) = |f_z(z)| - |f_{\overline{z}}(z)|$$

and

$$J_f(z) = |f_z(z)|^2 - |f_{\overline{z}}(z)|^2$$

whenever the partial derivatives of f exist. It follows that

$$K(f) = K_O(f) = K_I(f) = \frac{1+k}{1-k}$$

where

$$k := \operatorname*{ess\,sup}_{z \in \Omega} \frac{f_{\overline{z}}(z)}{f_z(z)}.$$

The 1-regular maps are precisely the holomorphic (or meromorphic) functions.

A major result in the theory, which has found important applications in complex dynamics, is the following theorem; see [3, Chapter 5] and [45, Chapter V].

Theorem 3.3. *Let $\mu : \overline{\mathbb{C}} \to \mathbb{C}$ be a measurable function with $k := \|\mu\|_\infty < 1$ and put $K := (1+k)/(1-k)$. Then there exists a K-quasiconformal homeomorphism $\phi : \overline{\mathbb{C}} \to \overline{\mathbb{C}}$ such that*

$$\frac{\phi_{\overline{z}}(z)}{\phi_z(z)} = \mu(z) \quad a.e. \tag{3.3}$$

The map ϕ may be chosen to fix 0, 1 and ∞, and with this normalization it is unique.

The equation (3.3) is called the *Beltrami equation*. A simple consequence of Theorem 3.3 is the following result.

Theorem 3.4. *Let $U, V \subset \mathbb{C}$ be simply connected domains, $U, V \neq \mathbb{C}$. Let $\mu : U \to \mathbb{C}$ be measurable with $k := \|\mu\|_\infty < 1$ and put $K := (1+k)/(1-k)$. Then there there exists a K-quasiconformal homeomorphism $\phi : U \to V$ such that $\phi_{\overline{z}}(z)/\phi_z(z) = \mu(z)$ a.e.*

Note that the case $\mu(z) \equiv 0$ is just the Riemann mapping theorem. Therefore Theorems 3.3 and 3.4 are also called the *measurable Riemann mapping theorem*. Applying these results to $\mu = f_{\overline{z}}/f_z$ for a quasiregular map f leads to the following result.

Theorem 3.5. *Let $U \subset \overline{\mathbb{C}}$ be a domain and let $f : U \to \overline{\mathbb{C}}$ be quasiregular. Then there exists a quasiconformal homeomorphism $\phi : U \to U$ with $K(\phi) = K(f)$ and a meromorphic function $g : U \to \overline{\mathbb{C}}$ such that $f = g \circ \phi$.*

4. Uniformly quasiregular maps

4.1. **Uniformly quasiregular maps in dimension two.** As mentioned in section 2, the proofs of many of the basic results in complex dynamics are based on Montel's theorem. Now Theorem 3.2 gives an analogue of Montel's theorem for quasiregular maps. However, in order to apply this result to the family $\{f^n\}$ for a quasiregular map f, one has to assume that all iterates f^n are K-quasiregular with the same K. Quasiregular maps with this property are called *uniformly quasiregular*.

In dimension 2 examples are given by maps of the form $\phi \circ g \circ \phi^{-1}$ where g is rational and where ϕ is a quasiconformal self-map of the Riemann sphere. It was shown in [31, 35, 44, 75] that every uniformly quasiregular self-map of the Riemann sphere has this form. In other words, a uniformly quasiregular self-map of the Riemann sphere is quasiconformally conjugate to a rational map, and thus studying the dynamics of uniformly quasiregular self-maps of the Riemann sphere reduces to studying dynamics of rational functions. An analogous remark applies to uniformly quasiregular self-maps of the plane.

4.2. **Existence of uniformly quasiregular maps in higher dimensions.** The results discussed in the previous section show that the dynamics of uniformly quasiregular maps are of interest only in dimension greater than 2. However, it is non-trivial that such maps exist at all. The first example of a uniformly quasiregular self-map of $\overline{\mathbb{R}^d}$ where $d \geq 3$ was given by Iwaniec and Martin [39].

Example 4.1. Let $0 < \delta < \pi/2$, define $h : [0, \pi] \to \mathbb{R}$,

$$h(\theta) = \begin{cases} \theta & \text{if } 0 \le \theta \le \delta, \\ \dfrac{2(\pi - \delta) - \pi\delta}{\pi - 2\delta} & \text{if } \delta < \theta \le \pi - \delta, \\ \theta + \pi & \text{if } \pi - \delta < \theta \le \pi, \end{cases}$$

and extend h to an odd map $h : [-\pi, \pi] \to \mathbb{R}$. Next the map $g : \mathbb{R}^3 \to \mathbb{R}^3$ is defined in cylindrical coordinates by $(r, \theta, x_3) \mapsto (2r, h(\theta), 2x_3)$.

In the domain $W_1 := \{(r, \theta, x_3) : |\theta| < \delta\}$ the map g is just scaling with a factor 2. In other words, in cartesian coordinates the map $g|_{W_1}$ takes the form $(x_1, x_2, x_3) \mapsto (2x_1, 2x_2, 2x_3)$. In $W_2 := \{(r, \theta, x_3) : |\theta - \pi| < \delta\}$ the map g consists of a rotation by the angle π around the x_3-axis, followed by scaling with a factor 2. We note that g is conformal in $W := W_1 \cup W_2$. In $\mathbb{R}^3 \setminus W$ the map g is essentially the winding map with

$$N = \frac{2(\pi - \delta) - \pi\delta}{\pi - 2\delta},$$

followed by scaling with a factor 2. (In contrast to the winding map, N need not be an integer here.)

It follows that g is quasiregular. In fact, we have $K(g) = N^2 \to 4$ as $\delta \to 0$. The map g extends to a quasiregular self-map of $\overline{\mathbb{R}^3}$ by putting $g(\infty) = \infty$.

We note that

$$g(W) = W_1 \quad \text{and} \quad g\left(\overline{\mathbb{R}^3} \setminus W\right) = \overline{\mathbb{R}^3} \setminus W_1.$$

Let now $a := (a_1, 0, 0)$ where $a_1 > 0$ is chosen so large that the closure of of the ball $V := \{x \in \mathbb{R}^3 : |x - a| < 1\}$ is contained in W_1 and that $g(V) \cap V = \emptyset$. Let ϕ be the conformal inversion at ∂V; that is, $\phi := \phi_1 \circ \phi_2$ where

$$\phi_1(x_1, x_2, x_3) = (x_1, x_2, -x_3) \quad \text{and} \quad \phi_2(x) = a + \frac{x - a}{|x - a|^2}.$$

Thus $\phi\left(\overline{\mathbb{R}^3} \setminus \overline{V}\right) \subset V$.

Finally we put $f = \phi \circ g$. Since ϕ is conformal we see that f is conformal in W. We also have

$$f(V) = \phi(g(V)) \subset \phi\left(W_1 \setminus \overline{V}\right) \subset V$$

and

$$f\left(\overline{\mathbb{R}^3} \setminus W\right) = \phi\left(g\left(\overline{\mathbb{R}^3} \setminus W\right)\right) \subset \phi\left(\overline{\mathbb{R}^3} \setminus W_1\right) \subset V.$$

To summarize, the domains V and W are such that

$$f(V) \subset V \subset W, \quad f \text{ is conformal in } W \quad \text{and} \quad f\left(\overline{\mathbb{R}^3} \setminus W\right) \subset V. \tag{4.1}$$

We see that every orbit of f intersects the part where f is not conformal at most once. It is not difficult to see that this implies that f is indeed uniformly quasiregular, with $K(f^n) \leq K(g)$ for all $n \in \mathbb{N}$.

The construction in Example 4.1 extends without difficulty to dimensions greater than 3. If V and W are as in (4.1), then the domain V is called a *conformal trap.* Martin [46] used conformal traps to show that if $f : \overline{\mathbb{R}^d} \to \overline{\mathbb{R}^d}$ is quasiregular, then there exists a uniformly quasiregular map $g : \overline{\mathbb{R}^d} \to \overline{\mathbb{R}^d}$ which has the same branch set as f.

Another method of constructing uniformly quasiregular maps is given in the following example due to Mayer [51].

Example 4.2. Let Z be the Zorich map from Example 3.2 and let $p \in \mathbb{N}$, $p \geq 2$. For $x, y \in \mathbb{R}^3$ we have $Z(x) = Z(y)$ if and only if there exist integers m and n such that $x - y = (4m, 4n, 0)$. It follows that if $Z(x) = Z(y)$ for $x, y \in \mathbb{R}^3$, then $Z(px) = Z(py)$. Thus we can define $f : Z(\mathbb{R}^3) \to \mathbb{R}^3$ by $f(Z(x)) := Z(px)$. Now $Z(\mathbb{R}^3) = \mathbb{R}^3 \setminus \{0\}$, but it is not difficult to see that f can be extended to a continuous map $f : \overline{\mathbb{R}^3} \to \overline{\mathbb{R}^3}$. For $n \in \mathbb{N}$ we have $f^n(Z(x)) = Z(p^n x)$. Thus $f^n(x) = Z(p^n Z^{-1}(x))$ in the regions where a branch of Z^{-1} can be defined. From this it follows that f is indeed uniformly quasiregular, with $K(f^n) \leq K(Z)^2$ for all $n \in \mathbb{N}$.

It is easy to see that $f^n(Z(x_1, x_2, x_3)) \to \infty$ if $x_3 > 0$ while $f^n(Z(x_1, x_2, x_3)) \to 0$ if $x_3 < 0$. Thus $f^n(x) \to \infty$ for $|x| > 1$ while $f^n(x) \to 0$ for $|x| < 1$.

A similar construction can be made if Zorich's map is replaced by an analogue of the Weierstraß $\wp$-function [50]. Instead of a "square beam" now a cube is mapped onto the upper half-space, and the map is extended by reflections along faces of cubes. The maps obtained this way are analogues of the so-called Lattès maps that play an important role in rational dynamics; see [48, 51, 52] for more details about these quasiregular analogues of Lattès maps.

Further examples of uniformly quasiregular maps were given by Peltonen [60]. We also mention that Martin and Peltonen [49] proved that every quasiregular map $f : \overline{\mathbb{R}^d} \to \overline{\mathbb{R}^d}$ has a factorization $g \circ \phi$ where $\phi : \overline{\mathbb{R}^d} \to \overline{\mathbb{R}^d}$ is quasiconformal and $g : \overline{\mathbb{R}^d} \to \overline{\mathbb{R}^d}$ is uniformly quasiregular. This can be seen as an extension of Theorem 3.5 to higher dimensions.

So there are a number of examples of interesting uniformly quasiregular self-maps of $\overline{\mathbb{R}^d}$. However, for $d \geq 3$ no examples of uniformly quasiregular self-maps of $\mathbb{R}^d$ which do not extend to uniformly quasiregular self-maps of $\overline{\mathbb{R}^d}$ are known.

4.3. **The dynamics of uniformly quasiregular maps.** In this section we briefly describe the dynamics of uniformly quasiregular self-maps of $\overline{\mathbb{R}^d}$. Good surveys can be found in the book by Iwaniec and Martin [40, Chapter 21] and the thesis by Siebert [71, Chapter 4], and we refer to them and the papers cited below for more details.

As in the case of rational functions one restricts to maps of degree at least 2. Here the degree $\deg(f)$ of a (not necessarily uniformly) quasiregular map $f : \overline{\mathbb{R}^d} \to \overline{\mathbb{R}^d}$ is defined as the maximal cardinality of the preimage of a point; that is,

$$\deg(f) := \max_{x\in\overline{\mathbb{R}^d}} |f^{-1}(x)|.$$

The *Fatou set* and *Julia set* are defined exactly as before:

$$F(f) := \left\{x \in \overline{\mathbb{R}^d} : \left\{f^k\right\} \text{ is normal at } x\right\} \quad \text{and} \quad J(f) := \overline{\mathbb{R}^d} \setminus F(f).$$

We mention that for the function from Example 4.1 the Julia set is a Cantor set while for the function f from Example 4.2 we have $J(f) = \{x \in \overline{\mathbb{R}^d} : |x| = 1\}$. For the function f obtained by replacing the Zorich map in the construction in Example 4.2 by a quasiregular analogue of the Weierstraß $\wp$-function we have $J(f) = \overline{\mathbb{R}^d}$.

The definition of orbits and periodic points can be carried over literally from section 2 to the quasiregular setting. However, since quasiregular maps need not be differentiable, we cannot define the multiplier of a periodic point. Consequently, the definition of attracting and repelling periodic points has to be modified. Of course, it suffices to consider the case of fixed points. First, a fixed point ξ of f is called *superattracting* if $\xi \in B_f$. It turns out that then there is indeed a neighborhood of ξ where the iterates of f tend to ξ.

Now we consider fixed points which do not lie in the branch set and which thus have a neighborhood where the function is injective. The following "topological" definition was given by Hinkkanen and Martin [36]: a fixed point ξ of f is called *attracting* if there exists a neighborhood U of ξ such that f is injective in U and $\overline{f(U)} \subset U$. If $f(U) \supset \overline{U}$ for some neighborhood U of ξ where f is injective, then ξ is called *repelling*. A more "analytic" definition was given in a joint paper by the same authors with Mayer [38]. It is shown first that if ξ

is a fixed point of a uniformly quasiregular map f, then f is lipschitz-continuous at ξ. Suppose for simplicity that $\xi = 0$. The *infinitesimal space* $\mathcal{D}f(\xi)$ is then defined as the set of all functions φ of the form $\varphi(x) = \lim_{k\to\infty} \lambda_k f(x/\lambda_k)$ where $\lambda_k \to \infty$. It then follows that the elements of $\mathcal{D}f(\xi)$ are quasiconformal homeomorphisms of $\mathbb{R}^d$ fixing 0. Moreover, if $\varphi^n \to 0$ locally uniformly for some $\varphi \in \mathcal{D}f(\xi)$, then $\varphi^n \to 0$ for all $\varphi \in \mathcal{D}f(\xi)$. Similarly $\varphi^{-n} \to 0$ for all $\varphi \in \mathcal{D}f(\xi)$ if this is the case for some $\varphi \in \mathcal{D}f(\xi)$. In the first case ξ is called *attracting* and in the second case ξ is called *repelling*. It is shown in [38] that the topological and the analytic definition are equivalent.

If f is differentiable at ξ, then $\mathcal{D}f(\xi)$ consists only of the derivative $Df(\xi)$. In this case, ξ is attracting if all eigenvalues have modulus less than 1 and repelling if all eigenvalues have modulus greater than 1. Note that a general quasiregular map f may also have saddle points, meaning that there are eigenvalues of modulus greater than 1 as well as eigenvalues of modulus less than 1. This does not happen for uniformly quasiregular maps; cf. [38, p. 92].

For further discussion of the dynamics of uniformly quasiregular maps near fixed points we refer, besides [38], also to [53].

It turns out that with the above definitions the conclusion of Theorem 2.1 holds almost literally, with $\overline{\mathbb{C}}$ of course replaced by $\overline{\mathbb{R}^d}$. Only (v) and (viii) require a small change: we have $|E(f)| \leq q$ where q is the constant from Rickman's theorem, and $X \supset J(f)$ holds for closed and completely invariant sets X with at least $q+1$ elements. We note, however, that these changes are required only by the proof. Possibly (v) and (viii) hold for uniformly quasiregular maps exactly as stated in Theorem 2.1.

It is open whether Theorem 2.2 holds for uniformly quasiregular maps, that is, whether the Julia set is the closure of the set of repelling periodic points. However, there are some partial results. Siebert [71, Satz 4.3.4] showed that every point in the Julia set is a limit point of periodic points. She also obtained the conclusion of Theorem 2.2 under an additional assumption on the branch set [71, Satz 4.3.6].

Theorem 4.1. *Let $f : \overline{\mathbb{R}^d} \to \overline{\mathbb{R}^d}$ be uniformly quasiregular. Suppose that*

$$J(f) \not\subset \overline{\bigcup_{k=1}^{\infty} f^k(B_f)}.$$

Then $J(f)$ is the closure of the set of repelling periodic points of f.

Even though it is not known whether the conclusion of Theorem 2.2 holds for all uniformly quasiregular maps, some results proved for rational functions using Theorem 2.2 have been extended to uniformly quasiregular self-maps of $\overline{\mathbb{R}^d}$ by different arguments. In particular, Hinkkanen, Martin and Mayer [38, Corollary 3.3] have shown that no subsequence of the iterates is normal in any open set intersecting the Julia set. Also, the conclusion of Theorem 2.3 holds literally for uniformly quasiregular maps [71, Lemma 4.1.9].

There is also a classification of periodic components of the Fatou set which is analogous to Theorem 2.4; see [38, Proposition 4.9]. However, it is not known whether rotation domains actually exist; cf. [37, 47].

There are a number of further interesting results on the dynamics of uniformly quasiregular maps. We refer the reader to the papers cited above for more details.

5. Non-uniformly quasiregular maps

5.1. **Dynamics on the Riemann sphere.** As before we define the Fatou set $F(f)$ of a quasiregular map $f:\overline{\mathbb{C}}\to\overline{\mathbb{C}}$ by

$$F(f) := \left\{z\in\overline{\mathbb{C}} : \left\{f^k\right\} \text{ is normal at } z\right\}.$$

Before proceeding further we consider an example. Here and in the following we denote by $D(c,r)$ the open disk of radius r around a point c. Disks with respect to the spherical metric χ will be denoted by $D_\chi(c,r)$. We also put $\mathbb{D} := D(0,1)$.

Example 5.1. Define $f:\overline{\mathbb{C}}\to\overline{\mathbb{C}}$ by $f(0)=0$, $f(\infty)=\infty$ and $f(re^{it}) = re^{i2t}$ for $0<r<\infty$ and $-\pi<t\le\pi$. In other words, f is the winding map of degree 2 in dimension 2. Thus f is 2-quasiregular.

We have $f^n(re^{it}) = re^{i2^nt}$ for $n\in\mathbb{N}$ and this implies that $F(f) = \emptyset$. On the other hand, it is easily seen that $\partial D(0,r)$ is closed and completely invariant for all $r>0$, that $O(D(0,r)) = D(0,r)$ for all $r>0$ and that $\overline{O^-(z)} = \partial D(0,|z|)$ for all $z\in\mathbb{C}\setminus\{0\}$.

This example shows that it does not seem to make sense to define the Julia set of f as the complement of $F(f)$, since then many of the basic properties of Julia sets such as (v), (vii) and (ix) of Theorem 2.1 would not hold.

A solution of this problem was found by Sun and Yang [79, 80, 81] who instead used (vii) to define $J(f)$. Following them we thus denote by $J(f)$ the set of all $z\in\overline{\mathbb{C}}$ such that if U is a neighborhood of z, then $|\overline{\mathbb{C}}\setminus O^+(U)|\le 2$. It then follows again that $\overline{\mathbb{C}}\setminus O^+(U)\subset E(f)$. We also define

$$Q(f) := \overline{\mathbb{C}}\setminus(F(f)\cup J(f)).$$

Sun and Yang called $\overline{\mathbb{C}} \setminus J(f) = Q(f) \cap F(f)$ the *quasinormal set*. We note, however, that the term "quasinormal" has a different meaning in the theory of normal families [19, 67].

In Example 5.1, which is also due to Sun and Yang, we have $F(f) = J(f) = \emptyset$ and $Q(f) = \overline{\mathbb{C}}$, so the concepts introduced do not give anything interesting here. The decisive insight of Sun and Yang was that this changes if $\deg(f) > K(f)$.

Theorem 5.1. *Let $f : \overline{\mathbb{C}} \to \overline{\mathbb{C}}$ be quasiregular. Suppose that* $\deg(f) > K(f)$. *Then $J(f) \neq \emptyset$; that is, there exists $z \in \overline{\mathbb{C}}$ such that $|\overline{\mathbb{C}} \setminus O^+(U)| \leq 2$ for every neighborhood U of z.*

Note that in Example 5.1 we have $\deg(f) = K(f) = 2$. This shows that the hypothesis $\deg(f) > K(f)$ cannot be weakened.

Because of the importance of this result, and since it does not seem to have been translated from the Chinese so far, we include a proof of Theorem 5.1 in section 5.2.

Sun and Yang showed that for functions satisfying the hypothesis of Theorem 5.1 many of the basic properties of $J(f)$ remain valid. Some of their results are summarized in the following theorem.

Theorem 5.2. *Let $f : \overline{\mathbb{C}} \to \overline{\mathbb{C}}$ be quasiregular. Suppose that* $\deg(f) > K(f)$. *Then properties* (i)–(v) *and* (vii)–(ix) *of Theorem* 2.1 *hold.*

Specifically, (i) is [80, Theorem 6], (ii) is [80, Theorem 8], (iii) is [80, Theorem 7], (iv) is [80, Theorem 12] and (v) follows easily from (i), (iii) and the definition of $J(f)$. Furthermore, (vii) holds now by definition, (viii) is [80, Theorem 5] and (ix) is [80, Theorem 9 (b)].

In order to discuss property (vi), we need to define attracting fixed points. We first consider an example.

Example 5.2. Let $\delta > 0$, define $g : [1, 3] \to \mathbb{R}$ by

$$g(r) = \begin{cases} 1 - \delta(r-1) & \text{if } 1 \leq r \leq 2, \\ 1 + \delta(r-3) & \text{if } 2 < r \leq 3, \end{cases}$$

and define $f : \overline{\mathbb{C}} \to \overline{\mathbb{C}}$ by

$$f(z) = \begin{cases} z & \text{if } |z| \leq 1, \\ z\, g(|z|) & \text{if } 1 < |z| \leq 3, \\ z + \delta(|z| - 3) z^2 & \text{if } 3 < |z| \leq 4, \\ z + \delta z^2 & \text{if } 4 < |z| < \infty, \\ \infty & \text{if } z = \infty. \end{cases}$$

If δ is small enough, then f is a quasiregular map of degree 2, and $K(f) \to 1$ as $\delta \to 0$. In particular, we have $K(f) < 2 = \deg(f)$ for small δ.

For $1 < r < 3$ we find that $\overline{f(D(0,r))} \subset D(0,r)$ and that f is injective in $D(0,r)$. Thus every point in $\overline{\mathbb{D}}$ is an attracting fixed point according to the topological definition of section 4.3.

This example suggests that the topological definition of attracting fixed points given in section 4.3 is not appropriate in the present context. The analytic definition cannot be used either, since quasiregular maps need not be lipschitz-continuous at fixed points.

Instead the following definition was used by Sun and Yang [81, Definition 4]. A fixed point ξ of a function $f : \overline{\mathbb{C}} \to \overline{\mathbb{C}}$ is called *attracting* if there exists $c \in (0,1)$ and a neighborhood U of ξ such that $\chi(f(z), f(\xi)) < c\,\chi(z,\xi)$ for all $z \in U$. (Recall here that $\chi(\cdot,\cdot)$ denotes the spherical metric.) Similarly we say that ξ is *repelling* if $\chi(f(z), f(\xi)) > c\,\chi(z,\xi)$ for some $c > 1$ and all z in some neighborhood U.

With this definition we find that if f is as in Theorem 5.2 and if ξ is an attracting fixed point, then $J(f) \subset \partial A(\xi)$. In fact, this follows from property (v) since $\partial A(\xi)$ is closed and completely invariant. However, as shown by the following example, we may have $\partial A(\xi) \not\subset J(f)$; that is, property (vi) of Theorem 2.1 need not hold.

Example 5.3. Let $\delta > 0$, define $g : [0,4] \to \mathbb{R}$ by

$$g(r) = \begin{cases} (1-\delta) & \text{if } 0 \le r \le 1, \\ 1 + \delta(r-2) & \text{if } 1 < r \le 2, \\ 1 - \delta(r-2) & \text{if } 2 < r \le 3, \\ 1 + \delta(r-4) & \text{if } 3 < r \le 4, \end{cases}$$

and define $f : \overline{\mathbb{C}} \to \overline{\mathbb{C}}$ by

$$f(z) = \begin{cases} z\,g(|z|) & \text{if } |z| \le 4, \\ z + \delta(|z|-4)z^2 & \text{if } 4 < |z| \le 5, \\ z + \delta z^2 & \text{if } 5 < |z| < \infty, \\ \infty & \text{if } z = \infty. \end{cases}$$

As in Example 5.2 we see that if δ is small enough, then f is a quasiregular map satisfying $K(f) < 2 = \deg(f)$.

It is not difficult to see that 0 is an attracting fixed point and that we have $D(0,2) \subset A(0)$ and $\partial D(0,2) \subset \partial A(0)$. On the other hand, $f(D(0,4)) = D(0,4)$ and thus $\partial D(0,2) \cap J(f) = \emptyset$. We conclude that $\partial D(0,2) \subset Q(f)$. Thus $\partial A(0) \not\subset J(f)$.

The conclusion of Theorem 2.2 also does not hold under the hypothesis of Theorem 5.1; that is, the repelling periodic points need not be dense in the Julia set. This is shown by the following example.

Example 5.4. Let $d \in \mathbb{N}$ and $\delta > 0$. Define $f : \overline{\mathbb{C}} \to \overline{\mathbb{C}}$ by

$$f(z) = \begin{cases} z^2 & \text{if } |z| \le 1, \\ z^2 \left(1 + \dfrac{\sin(\pi|z|)}{\pi}\right) & \text{if } 1 < |z| \le 2, \\ z^2 + \delta(|z| - 2) z^d & \text{if } 2 < |z| \le 3, \\ z^2 + \delta z^d & \text{if } 3 < |z| < \infty, \\ \infty & \text{if } z = \infty. \end{cases}$$

For small δ and large d the map f is quasiregular and $K(f) < d = \deg(f)$. It can be shown that $\partial\mathbb{D} \subset J(f)$. However, $\partial\mathbb{D}$ does not contain repelling periodic points, since

$$\lim_{r \to 1+} \left| \frac{f(re^{it}) - f(e^{it})}{re^{it} - e^{it}} \right| = 1$$

for $-\pi < t \le \pi$.

While the Julia set thus need not be the closure of the set of repelling periodic points, there is a relation between the Julia set and periodic points. In particular we note that Sun and Yang (see [79, Theorem 5.1] or [80, Theorem 3 (i)]) showed that – under the hypotheses of Theorem 5.1 – every point in $J(f)$ is a limit point of periodic points. We mention that Fatou, in his proof of Theorem 2.2, first proved that every point in the Julia set is a limit point of periodic points, and then proved that there are only finitely many non-repelling periodic points. As shown by the above example, the latter result does not hold in the present context.

Sun and Yang also obtained an analogue of Theorem 2.2 using repelling periodic *sets*; cf. [80, Theorem 14]. A further result [80, Theorem 16] says that the conclusion of Theorem 2.2 holds if the partial derivatives of f exist and are continuous, except possibly for finitely many points.

They also showed [80, Theorem 15 (b)] that Theorem 2.3, which for rational functions is usually proved using Theorem 2.2, holds for all functions satisfying the hypothesis of Theorem 5.1.

On the other hand, the classification of invariant Fatou components given in Theorem 2.4 does not extend to this setting. In fact, let f be the function from Example 5.3. Then $U := \{z : 2 < |z| < 4\}$ is an invariant component of $F(f)$, with $f^n(z) \to 2z/|z|$ for $z \in U$ as $n \to \infty$. Thus U is of none of the types described in Theorem 2.4.

However, various other results of complex dynamics can be extended to quasiregular maps satisfying the hypothesis of Theorem 5.1. Here we only mention that Sun and Yang obtained results about the box dimension of the Julia set [81, Theorem 4], about the number of components of the quasinormal set, and about completely invariant components of the quasinormal set [81, Section 3]. We refer the reader to the papers by Sun and Yang [79, 80, 81] for further discussion.

5.2. **Proof of Theorem 5.1.** The main tool in the proof is a lemma about K-quasiregular maps that omit three values (Lemma 5.3 below). Sun and Yang based their proof of this lemma on results about the value distribution theory of plane quasiregular maps that they developed earlier; see [77, 78] and also their later paper [82]. In our proof of this lemma we will use the existence theorem for solutions of the Beltrami equation (i.e., Theorems 3.3–3.5) to reduce the case of quasiregular maps to that of meromorphic functions.

Before we can begin with the proof, we need some further lemmas. The first lemma is a direct consequence of Montel's theorem and Marty's theorem [32, Theorem 6.3]. Here $f^{\#} := |f'|/(1+|f|^2)$ denotes the spherical derivative of a meromorphic function f.

Lemma 5.1. *Let $a_1, a_2, a_3 \in \overline{\mathbb{C}}$ be distinct. Then there exists $M > 0$ such that if $c \in \mathbb{C}$, $r > 0$ and f is meromorphic in $D(c,r)$ with $f(z) \neq a_j$ for all $z \in D(c,r)$ and $j \in \{1,2,3\}$, then $f^{\#}(c) \leq M/r$.*

We denote the modulus of a doubly connected domain U by $\operatorname{mod}(U)$. We thus have

$$\operatorname{mod}\left(\{z \in \mathbb{C} : r_1 < |z| < r_2\}\right) = \frac{1}{2\pi} \log\left(\frac{r_2}{r_1}\right)$$

if $0 < r_1 < r_2 < \infty$. In the following lemma we put $1/(\log(1/x)) = 0$ if $x = 0$.

Lemma 5.2. *Let E be a compact, connected subset of $\mathbb{D}$ containing 0. Define*

$$h : \mathbb{R} \to [0,1), \quad h(\theta) = \max\{r : re^{i\theta} \in E\}.$$

Then

$$\int_{-\pi}^{\pi} \frac{1}{\log \frac{1}{h(\theta)}} d\theta \leq \frac{1}{\operatorname{mod}\left(\mathbb{D}\backslash E\right)}.$$

Proof. We may assume that the integral on the left hand side is positive since otherwise the conclusion is trivial. Let Γ be the set of all closed curves in $\mathbb{D}\backslash E$ that separate the two boundary components. It is well-known [2, p. 53] that the extremal length $\lambda_{\mathbb{D}\backslash E}(\Gamma)$ of this curve family satisfies

$$\lambda_{\mathbb{D}\backslash E}(\Gamma) = \frac{1}{\operatorname{mod}(\mathbb{D}\backslash E)}. \tag{5.1}$$

Define $\rho : \mathbb{D}\backslash E \to \mathbb{R}$ by

$$\rho(re^{i\theta}) = \frac{1}{r \log \frac{1}{h(\theta)}}$$

for $-\pi < \theta \leq \pi$ and $h(\theta) < r < 1$ and put $\rho(z) = 0$ otherwise. By the definition of extremal length we have

$$\lambda_{\mathbb{D}\backslash E}(\Gamma) \geq \frac{L(\Gamma,\rho)^2}{A(\mathbb{D}\backslash E,\rho)} \tag{5.2}$$

where

$$L(\Gamma,\rho)^2 = \inf_{\gamma\in\Gamma} \int_\gamma \rho(z)|dz| \quad \text{and} \quad A(\mathbb{D}\backslash E,\rho) = \iint_{\mathbb{D}\backslash E} \rho(z)^2\, dx\, dy.$$

Let now $\gamma : [0,1] \to \mathbb{D}\backslash E$, $\gamma(t) = r(t)e^{i\varphi(t)}$ be a parametrization of a curve $\gamma \in \Gamma$. We may assume that γ surrounds 0 once in the positive direction so that $\varphi(1) - \varphi(0) = 2\pi$. We obtain

$$\int_\gamma \rho(z)|dz| = \int_0^1 \frac{1}{\log \frac{1}{h(\varphi(t))}} \frac{|d\gamma(t)|}{r(t)} \geq \int_0^1 \frac{1}{\log \frac{1}{h(\varphi(t))}} d\varphi(t) = \int_{\varphi(0)}^{\varphi(1)} \frac{1}{\log \frac{1}{h(\theta)}} d\theta.$$

Since h is 2π-periodic we thus have

$$L(\Gamma,\rho) \geq \int_{-\pi}^{\pi} \frac{1}{\log \frac{1}{h(\theta)}} d\theta. \tag{5.3}$$

On the other hand,

$$\begin{aligned} A(\mathbb{D}\backslash E,\rho) &= \iint_{\mathbb{D}\backslash E} \rho(z)^2\, dx\, dy \\ &= \int_{-\pi}^{\pi} \int_{h(\theta)}^{1} \left(\frac{1}{r \log \frac{1}{h(\theta)}} \right)^2 r\, dr\, d\theta \\ &= \int_{-\pi}^{\pi} \frac{1}{\log \frac{1}{h(\theta)}} d\theta. \end{aligned} \tag{5.4}$$

The conclusion now follows from (5.1), (5.2), (5.3) and (5.4). □

For a domain U and a quasiregular map $f : U \to \overline{\mathbb{C}}$ we put

$$S(U,f) := \frac{1}{\pi} \int_U \frac{J_f(z)}{(1+|f(z)|^2)^2} dx\, dy.$$

Then $S(U,f)$ is the area of the image of U under f on the Riemann sphere, counted according to multiplicity, and divided by the area of the Riemann sphere for normalization. Thus $S(U,f)$ is the average number with which the points on the Riemann sphere are taken by f in U.

If f is meromorphic, then $J_f(z) = |f'(z)|^2$ and thus

$$S(U,f) = \frac{1}{\pi}\int_U f^{\#}(z)^2\, dx\, dy. \tag{5.5}$$

Lemma 5.3. *Let $a_1, a_2, a_3 \in \overline{\mathbb{C}}$ be distinct. Then there exists a constant C such that if f is quasiregular in a disk $D(c,r)$ and if $f(z) \neq a_j$ for all $z \in D(c,r)$ and $j \in \{1,2,3\}$, then $S\left(D\left(c, \frac{1}{2}r\right), f\right) \leq C\, K(f)$.*

As mentioned, Lemma 5.3 is due to Sun and Yang; see [78, equation (4.1)] and also [82, p. 441]. They show that with $\delta := \min_{j\neq k} \chi(a_j, a_k)$ the conclusion of Lemma 5.3 holds with $C := 2^{15}\pi^2/(\delta^6 \log 2)$.

Proof of Lemma 5.3. Without loss of generality we may assume that $c = 0$ and $r = 1$. By Theorem 3.5 there exists a quasiconformal homeomorphism $\phi : \mathbb{D} \to \mathbb{D}$ with $\phi(0) = 0$ and a meromorphic function $g : \mathbb{D} \to \overline{\mathbb{C}}$ such that $f = g \circ \phi$. Clearly, we also have $g(z) \neq a_j$ for all $z \in \mathbb{D}$ and $j \in \{1,2,3\}$. Using Lemma 5.1 and (5.5) we obtain

$$\begin{aligned} S\left(D\left(0, \tfrac{1}{2}\right), f\right) &= S\left(\phi\left(D\left(0, \tfrac{1}{2}\right)\right), g\right) \\ &= \frac{1}{\pi}\int_{\phi\left(D\left(0,\frac{1}{2}\right)\right)} g^{\#}(z)^2\, dx\, dy \\ &\leq \frac{M^2}{\pi}\int_{\phi\left(D\left(0,\frac{1}{2}\right)\right)} \frac{1}{(1-|z|)^2} dx\, dy \end{aligned}$$

for some constant M depending only on the a_j. With $E := \overline{\phi\left(D\left(0, \frac{1}{2}\right)\right)}$ we deduce from Lemma 5.2, using also that $\log(1/x) \leq 1/x - 1 =$

$(1-x)/x$ for $0<x<1$, that

$$\begin{aligned}\int_{\phi\left(D\left(0,\frac{1}{2}\right)\right)} \frac{1}{(1-|z|)^2} dx\, dy &\leq \int_{-\pi}^{\pi} \int_0^{h(\theta)} \frac{r}{(1-r)^2} dr\, d\theta \\ &= \int_{-\pi}^{\pi} \left(\frac{h(\theta)}{1-h(\theta)} + \log(1-h(\theta)) \right) d\theta \\ &\leq \int_{-\pi}^{\pi} \frac{h(\theta)}{1-h(\theta)} d\theta \\ &\leq \int_{-\pi}^{\pi} \frac{1}{\log \frac{1}{h(\theta)}} d\theta \\ &\leq \frac{1}{\operatorname{mod}(\mathbb{D}\backslash E)}.\end{aligned}$$

Noting that $\mathbb{D}\backslash E = \phi\left(\mathbb{D} \setminus \overline{D\left(0,\frac{1}{2}\right)}\right)$ we have [45, Theorem I.7.1]

$$\operatorname{mod}(\mathbb{D}\backslash E) \geq \frac{\operatorname{mod}\left(\mathbb{D} \setminus \overline{D\left(0,\frac{1}{2}\right)}\right)}{K(\phi)} = \frac{\log 2}{2\pi K(\phi)}.$$

Since $K(f)=K(\phi)$ we finally conclude from the last three inequalities that

$$S\left(D\left(0,\tfrac{1}{2}\right), f\right) \leq \frac{2M^2}{\log 2} K(f)$$

so that the conclusion follows with $C := 2M^2/\log 2$. □

We now begin with the actual proof of Theorem 5.1. We put $d := \deg(f)$ and note first that $S\left(\overline{\mathbb{C}}, f^n\right) = \deg(f^n) = d^n$ for $n \in \mathbb{N}$.

Next we note that there exists an absolute constant L such that if $n \in \mathbb{N}$, then $\overline{\mathbb{C}}$ can be covered by Ln^2 spherical disks of radius $1/n$. We thus find that for $n \in \mathbb{N}$ there exists $c_n \in \overline{\mathbb{C}}$ such that

$$S\left(D_\chi\left(c_n, \frac{1}{n}\right), f^n\right) \geq \frac{1}{Ln^2} S\left(\overline{\mathbb{C}}, f^n\right) = \frac{d^n}{Ln^2}.$$

The sequence (c_n) has a convergent subsequence, say $c_{n_k} \to c$.

We shall show that $c \in J(f)$. Suppose that this is not the case. Without loss of generality we may assume that $c \neq \infty$. Then there exists $\delta > 0$ and distinct points $a_1, a_2, a_3 \in \overline{\mathbb{C}}$ such that $O^+(D(c, 2\delta)) \subset \overline{\mathbb{C}} \setminus \{a_1, a_2, a_3\}$. It follows from Lemma 5.3 that

$$S(D(c,\delta), f^n) \leq CK(f^n) \leq CK(f)^n.$$

On the other hand, for sufficiently large k we have $D_\chi(c_{n_k}, 1/n_k) \subset D(c,\delta)$ and thus

$$S\left(D(c,\delta), f^{n_k}\right) \geq S\left(D_\chi\left(c_{n_k}, \frac{1}{n_k}\right), f^{n_k}\right) \geq \frac{d^{n_k}}{Ln_k^2}.$$

Combining the last two inequalities we find that

$$\frac{d^{n_k}}{Ln_k^2} \leq CK(f)^{n_k},$$

which for large k contradicts the assumption that $d > K(f)$.

5.3. **Dynamics in higher dimensions.** The dynamics of non-uniformly quasiregular maps in higher dimensions have been studied only recently, and so far there seem to be only a few papers devoted to them. In the following, we describe the main results obtained in these papers.

Fletcher and Nicks [30] considered quasiregular maps $f : \mathbb{R}^d \to \mathbb{R}^d$ satisfying $f(x) \to \infty$ as $x \to \infty$ and which thus by putting $f(\infty) := \infty$ extend to quasiregular self-maps of $\overline{\mathbb{R}^d}$. Such maps are said to be of *polynomial type*; cf. [34]. We mention that Sun and Yang [80, Theorems 6–10] as well as Wu and Sun [83] discussed the dynamics of quasiregular maps of polynomial type in dimension 2.

The paper by Fletcher and Nicks [30] is mainly concerned with the escaping set of such maps in higher dimensions. As in Theorem 5.1 a lower bound for the degree is required. The following theorem summarizes some of their results.

Theorem 5.3. *Let $f : \mathbb{R}^d \to \mathbb{R}^d$ be quasiregular of polynomial type. Suppose that* $\deg(f) > K_I(f)$. *Then $I(f) \neq \emptyset$ and $\partial I(f)$ is perfect. Moreover, $I(f)$ is connected and $I(f)$, $\partial I(f)$ and $\overline{\mathbb{R}^d} \setminus \overline{I(f)}$ are completely invariant.*

As shown by the winding map (Example 3.1), the hypothesis $\deg(f) > K_I(f)$ cannot be weakened here.

Quasiregular maps $f : \mathbb{R}^d \to \mathbb{R}^d$ which are not of polynomial type have an essential singularity at ∞. The escaping set of such maps was studied in [15]. The main result of the paper says that some of the theorems of Eremenko [26] and Rippon and Stallard [65] that we mentioned in section 2.2 can be extended from entire functions to quasiregular maps.

Theorem 5.4. *Let $f : \mathbb{R}^d \to \mathbb{R}^d$ be a quasiregular map with an essential singularity at ∞. Then $I(f) \neq \emptyset$. Moreover, $I(f)$ has an unbounded component.*

On the other hand, it was also shown in [15] that Eremenko's result that $\overline{I(f)}$ does not have bounded components for entire f does not extend to the quasiregular setting.

The dynamics of Zorich maps were studied in [13]. More specifically, for $a > 0$ the map f_a given by $f_a(x) := Z(x) - (0, 0, a)$ was considered. The main result says that the results of Devaney and Krych [21], McMullen [54] and Karpińska [43] on the dynamics of exponential maps that we mentioned at the end of section 2.2 have analogues for Zorich maps. Specifically, the following result was proved in [13, Theorem 1].

Theorem 5.5. *If a is sufficiently large, then f_a has a unique attracting fixed point ξ, the set $J := \mathbb{R}^3 \setminus A(\xi)$ consists of uncountably many pairwise disjoint hairs, the set C of endpoints of these hairs has Hausdorff dimension 3, and $J \backslash C$ has Hausdorff dimension 1.*

A higher dimensional quasiregular analogue of the trigonometric functions was constructed in [14], and some of the results for trigonometric functions mentioned in section 2.2 were extended to this setting. In particular, the result of Schleicher [68] was extended to higher dimensions.

Theorem 5.6. *For each $d \geq 2$ there exists a representation of $\mathbb{R}^d$ as a union of hairs such that the intersection of any two hairs is either empty or consists of the common endpoint, and the union of the hairs without their endpoints has Hausdorff dimension 1.*

It turns out that in the quasiregular setting the proof is actually easier than in the case of entire functions, since the quasiregular map can be chosen to be uniformly expanding.

Finally we mention the following result of Siebert [71, 72] which extends a result from [9] (see also [7, Section 4]) for entire functions to quasiregular maps.

Theorem 5.7. *Let $f : \mathbb{R}^d \to \mathbb{R}^d$ be a quasiregular map with an essential singularity at ∞ and let $n \geq 2$. Then f has infinitely many periodic points of period n.*

Strictly speaking, this result is not about dynamics, since the iterates f^n are considered for fixed n, while dynamics is concerned with their behavior as n tends to ∞. However, it seems likely that Siebert's remarkable result will find applications in the dynamics of quasiregular maps.

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