NORMAL FAMILIES AND FIXED POINTS OF ITERATES

WALTER BERGWEILER

Dedicated to Professor Yang Lo on the occasion of his 70th birthday

Abstract. Let $\mathcal{F}$ be a family of holomorphic functions and suppose that there exists $\varepsilon > 0$ such that if $f \in \mathcal{F}$, then $|(f^2)'(\xi)| \leq 4 - \varepsilon$ for all fixed points ξ of the second iterate f^2. We show that then $\mathcal{F}$ is normal. This is deduced from a result which says that if p is a polynomial of degree at least 2, then p^2 has a fixed point ξ such that $|(p^2)'(\xi)| \geq 4$. The results are motivated by a problem posed by Yang Lo.

1. Introduction and main results

Yang Lo [14, Problem 8] posed the following problem in 1992.

Problem. Let $\mathcal{F}$ be a family of entire functions, let $D \subset \mathbb{C}$ be a domain and let $n \geq 2$ be a fixed integer. Suppose that for every $f \in \mathcal{F}$ the n-th iterate f^n does not have fixed points in D. Is $\mathcal{F}$ normal in D?

An affirmative answer was given by Essén and Wu [7] in 1998. They did not require that the functions in $\mathcal{F}$ are entire but only that they are holomorphic in D. The iterates $f^n : D_n \to \mathbb{C}$ of such a function $f : D \to \mathbb{C}$ are defined by $D_1 := D$, $f^1 := f$ and $D_n := f^{-1}(D_{n-1})$, $f^n := f^{n-1} \circ f$ for $n \in \mathbb{N}$, $n \geq 2$. Note that $D_2 = f^{-1}(D_1) \subset D = D_1$ and thus $D_{n+1} \subset D_n \subset D$ for all $n \in \mathbb{N}$.

In a subsequent paper, Essén and Wu [8, Theorem 1] gave the following generalization of their result. Here a fixed point ξ of f is called *repelling* if $|f'(\xi)| > 1$.

Theorem A. *Let $D \subset \mathbb{C}$ be a domain and let $\mathcal{F}$ be the family of all holomorphic functions $f : D \to \mathbb{C}$ for which there exists $n = n(f) > 1$ such that f^n has no repelling fixed point. Then $\mathcal{F}$ is normal.*

There are a number of further developments initiated by Yang Lo's question. For example, his question has also been considered for meromorphic [12] and quasiregular [11] functions. Other papers related to Yang Lo's problem include [1, 4, 5, 13]; see [3, section 3] for further discussion. The following result was proved in [2, Theorem 1.3].

Theorem B. *For each integer $n \geq 2$ there exists a constant $K_n > 1$ with the following property: if $D \subset \mathbb{C}$ is a domain and $\mathcal{F}$ is a family of holomorphic functions $f : D \to \mathbb{C}$ such that $|(f^n)'(\xi)| \leq K_n$ for all fixed points ξ of f^n, then $\mathcal{F}$ is normal.*

Considering the family $\mathcal{F} = \{az^2\}_{a \in \mathbb{C}\setminus\{0\}}$ we see that the conclusion of Theorem B does not hold for $K_n = 2^n$. The following conjecture says that it holds for $K_n < 2^n$.

Date: September 8, 2009.

1991 *Mathematics Subject Classification.* Primary 30D45; Secondary 30D05, 37F10.

Supported by the EU Research Training Network CODY, the ESF Networking Programme HCAA and the Deutsche Forschungsgemeinschaft, Be 1508/7-1.

Conjecture A. Let $D \subset \mathbb{C}$ be a domain, $n \geq 2$ and $\varepsilon > 0$. Let $\mathcal{F}$ be the family of all holomorphic functions $f : D \to \mathbb{C}$ such that $|(f^n)'(\xi)| \leq 2^n - \varepsilon$ for all fixed points ξ of f^n. Then $\mathcal{F}$ is normal.

We show that this conjecture is true for $n = 2$.

Theorem 1. *Let $D \subset \mathbb{C}$ be a domain and $\varepsilon > 0$. Let $\mathcal{F}$ be the family of all holomorphic functions $f : D \to \mathbb{C}$ such that $|(f^2)'(\xi)| \leq 4 - \varepsilon$ for all fixed points ξ of f^2. Then $\mathcal{F}$ is normal.*

We deduce Theorem 1 from a result about fixed points of iterated polynomials. In fact, we shall see that Conjecture A is equivalent to the following conjecture.

Conjecture B. Let p be a polynomial of degree at least 2 and let $n \geq 2$. Then p^n has a fixed point ξ satisfying $|(p^n)'(\xi)| \geq 2^n$.

The equivalence of these two conjectures is seen by the following result.

Theorem 2. *Let $n \geq 2$ and let $C_n > 0$ be such that for every polynomial p of degree at least 2 there exists a fixed point ξ of p^n such that $|(p^n)'(\xi)| \geq C_n$. Let $D \subset \mathbb{C}$ be a domain, let $\varepsilon > 0$ and let $\mathcal{F}$ be the family of all holomorphic functions $f : D \to \mathbb{C}$ such that $|(f^n)'(\xi)| \leq C_n - \varepsilon$ for all fixed points ξ of f^n. Then $\mathcal{F}$ is normal.*

Theorem 1 now follows from Theorem 2 and the following result which says that we can take $C_2 = 4$.

Theorem 3. *Let p be a polynomial of degree at least 2. Then p^2 has a fixed point ξ satisfying $|(p^2)'(\xi)| \geq 4$.*

We conclude this introduction with a conjecture which is stronger than Conjecture B.

Conjecture C. Let p be a polynomial of degree $d \geq 2$ and let $n \geq 2$. Then p^n has a fixed point ξ satisfying $|(p^n)'(\xi)| \geq d^n$.

The monomial $p(z) = z^d$ shows that this would be best possible. A. E. Eremenko and G. M. Levin [6, Theorem 3] have shown that if p is a polynomial of degree $d \geq 2$ which is not conjugate to the monomial z^d, then there exists $n \geq 2$ such that p^n has a fixed point ξ satisfying $|(p^n)'(\xi)| > d^n$.

2. Proof of Theorem 2

We shall use the following result proved in [2, Theorem 1.2].

Lemma 1. *Let f be a transcendental entire function and let $n \in \mathbb{N}$, $n \geq 2$. Then there exists a sequence (ξ_k) of fixed points of f^n such that $(f^n)'(\xi_k) \to \infty$ as $k \to \infty$.*

The other main tool in the proof of Theorem 2 is the following lemma due to X. Pang and L. Zalcman [10, Lemma 2].

Lemma 2. *Let $\mathcal{F}$ be a family of functions meromorphic on the unit disc, all of whose zeros have multiplicity at least k, and suppose that there exists $A \geq 1$ such that $|g^{(k)}(\xi)| \leq A$ whenever $g(\xi) = 0$, $g \in \mathcal{F}$. Then if $\mathcal{F}$ is not normal there exist, for each $0 \leq \alpha \leq k$, a number $r \in (0, 1)$, points $z_j \in D(0, r)$, functions $g_j \in \mathcal{F}$ and positive numbers ρ_j tending to zero such that*

$$\frac{g_j(z_j + \rho_j z)}{\rho_j^\alpha} \to G(z)$$

locally uniformly, where G is a nonconstant meromorphic function on $\mathbb{C}$ such that the spherical derivative $G^\#$ of G satisfies $G^\#(z) \le G^\#(0) = kA + 1$ for all $z \in \mathbb{C}$.

We shall only need the case $k = 1$ of Lemma 2. This special case can also be found in Pang's paper [9, Lemma 2]. The case $\alpha = 0$ is known as Zalcman's lemma [15, 16].

Proof of Theorem 2. We denote by $\mathcal{F}(D, n, K)$ the family of all functions f holomorphic in D such that $|(f^n)'(\xi)| \le K$ whenever $f^n(\xi) = \xi$. Note that this implies that $|f'(\xi)| \le \sqrt[n]{K}$ whenever $f(\xi) = \xi$. Suppose that the conclusion of Theorem 2 does not hold. Then there exist a domain $D \subset \mathbb{C}$, $n \ge 2$ and $\varepsilon > 0$ such that $\mathcal{F}(D, n, C_n - \varepsilon)$ is not normal. We may assume that D is the unit disk.

We choose a non-normal sequence (f_j) in $\mathcal{F}(D, n, C_n - \varepsilon)$. With $g_j(z) := f_j(z) - z$ we find that if $g_j(\xi) = 0$, then $f_j(\xi) = \xi$ and thus

$$|g_j'(\xi)| \le |f_j'(\xi)| + 1 \le \sqrt[n]{C_n - \varepsilon} + 1 =: A.$$

We may assume here that ε is chosen so small that $A^n > C_n$. Clearly, the sequence (g_j) is also not normal. Applying Lemma 2 with $\alpha = k = 1$ we may assume, passing to a subsequence if necessary, that there exist $z_j \in D$ and $\rho_j > 0$ such that $g_j(z_j + \rho_j z)/\rho_j \to G(z)$ for some entire function G satisfying $G^\#(z) \le G^\#(0) = A + 1$ for all $z \in \mathbb{C}$. With $L_j(z) = z_j + \rho_j z$ we find that

$$h_j(z) := L_j^{-1}(f_j(L_j(z))) = \frac{f_j(z_j + \rho_j z) - z_j}{\rho_j} = \frac{g_j(z_j + \rho_j z)}{\rho_j} + z \to G(z) + z.$$

With $F(z) := G(z) + z$ we thus have $h_j(z) \to F(z)$ as $j \to \infty$. It follows that $h_j^n(z) \to F^n(z)$. The assumption that $f_j \in \mathcal{F}(D, n, C_n - \varepsilon)$ implies that $h_j \in \mathcal{F}(L_j^{-1}(D), n, C_n - \varepsilon)$; that is, $|(h_j^n)'(\xi)| \le C_n - \varepsilon$ whenever $h_j^n(\xi) = \xi$. We deduce that $F \in \mathcal{F}(\mathbb{C}, n, C_n - \varepsilon)$. It follows from the definition of C_n that F cannot be a polynomial of degree greater than one. And Lemma 1 implies that F cannot be transcendental. Thus F is a polynomial of degree 1 at most. Now $|F'(0)| \ge |G'(0)| - 1 \ge G^\#(0) - 1 = A$. Hence F has the form $F(z) = az + b$ where $|a| \ge A$. With $\xi := b/(1 - a)$ we obtain $F(\xi) = \xi$ and $|F'(\xi)| = |a| \ge A$. Thus $F^n(\xi) = \xi$ and $|(F^n)'(\xi)| = |a^n| \ge A^n > C_n$, contradicting $F \in \mathcal{F}(\mathbb{C}, n, C_n - \varepsilon)$.

3. Proof of Theorem 3

The following lemma is due to A. E. Eremenko and G. M. Levin [6, Lemma 1].

Lemma 3. *Let p be a polynomial of degree $d \ge 2$. Then there exists c such that*

$$\sum_{\{z : p^n(z) = z\}} (p^n)'(z) = d^n(d^n - 1) + c^n$$

for all $n \in \mathbb{N}$.

In fact, they show that this holds with

$$c = \sum_{\{z : p(z) = w\}} p'(z),$$

for arbitrary $w \in \mathbb{C}$, but we do not need this result. We note that in the sum occuring in the lemma each fixed point of p^n is counted according to multiplicity.

Proof of Theorem 3. Suppose that p is a polynomial such that $|(p^2)'(\xi)| < 4$ for each fixed point ξ of p^2. Then $|p'(\xi)| < 2$ for each fixed point ξ of p. It follows from Lemma 3 that

$$|d(d-1)+c| < 2d \tag{1}$$

and

$$|d^2(d^2-1)+c^2| < 4d^2. \tag{2}$$

Now (1) yields $c = -d(d-1) + re^{it}$ where $0 \le r < 2d$ and $t \in \mathbb{R}$. Thus

$$c^2 = (-d(d-1)+re^{it})^2 = d^2(d-1)^2 - 2d(d-1)re^{it} + r^2 e^{i2t}$$

and hence

$$\begin{aligned}
\operatorname{Re}(c^2) &= d^2(d-1)^2 - 2d(d-1)r\cos t + r^2\cos 2t \\
&= d^2(d-1)^2 - 2d(d-1)r\cos t + r^2(2\cos^2 t - 1) \\
&= d^2(d-1)^2 - r^2 - 2d(d-1)r\cos t + 2r^2\cos^2 t \\
&= \frac{1}{2}d^2(d-1)^2 - r^2 + 2\left(\frac{1}{2}d(d-1) - r\cos t\right)^2 \\
&\ge \frac{1}{2}d^2(d-1)^2 - r^2 \\
&> \frac{1}{2}d^2(d-1)^2 - 4d^2.
\end{aligned}$$

Thus

$$\begin{aligned}
|d^2(d^2-1)+c^2| &\ge \operatorname{Re}(d^2(d^2-1)+c^2) \\
&> d^2(d^2-1) + \frac{1}{2}d^2(d-1)^2 - 4d^2 \\
&= d^2\left(\frac{3}{2}d^2 - d - \frac{9}{2}\right).
\end{aligned}$$

Combining this with (2) we find that

$$d^2\left(\frac{3}{2}d^2 - d - \frac{9}{2}\right) < 4d^2$$

and thus that

$$\frac{3}{2}d^2 - d - \frac{9}{2} < 4.$$

The last inequality can be rewritten as

$$\frac{3}{2}d^2 - d - \frac{17}{2} = \frac{3}{2}\left(\left(d - \frac{1}{3}\right)^2 - \frac{52}{9}\right) < 0.$$

It follows that $d < (1+2\sqrt{13})/3 < 3$.

It thus remains to consider the case $d = 2$. Then $c = 0$ in Lemma 3. Let ξ_1, ξ_2 be the fixed points of p and $\xi_1, \xi_2, \xi_3, \xi_4$ those of p^2. Then $p'(\xi_1) + p'(\xi_2) = 2$ so that $p'(\xi_{1,2}) = 1 \pm a$ for some $a \in \mathbb{C}$, with $|1 \pm a| < 2$. In particular, we have $|\operatorname{Re} a| < 1$. Moreover, Lemma 3 yields

$$12 = \sum_{j=1}^{4}(p^2)'(\xi_j) = (1+a)^2 + (1-a)^2 + (p^2)'(\xi_3) + (p^2)'(\xi_4)$$

and hence

$$(p^2)'(\xi_3) + (p^2)'(\xi_4) = 2(5 - a^2).$$

It follows that

$$\text{Re}\left((p^2)'(\xi_3) + (p^2)'(\xi_4)\right) = 2(5 - \text{Re}(a^2)) \geq 2(5 - (\text{Re}\, a)^2) > 8.$$

Hence there exists $j \in \{3, 4\}$ with $|(p^2)'(\xi_j)| > 4$, a contradiction.

References

[1] Detlef Bargmann and Walter Bergweiler, Periodic points and normal families. Proc. Amer. Math. Soc. **129** (2001), 2881–2888.

[2] Walter Bergweiler, Quasinormal families and periodic points. Complex analysis and dynamical systems II, 55–63, Contemp. Math. **382**, Amer. Math. Soc., Providence, RI, 2005.

[3] Walter Bergweiler, Bloch's principle. Comput. Methods Funct. Theory **6** (2006), 77–108.

[4] Jianming Chang and Mingliang Fang, Normal families and fixed points. J. Anal. Math. **95** (2005), 389–395.

[5] Jianming Chang, Mingliang Fang and Lawrence Zalcman, Normality and attracting fixed points. Bull. Lond. Math. Soc. **40** (2008), 777–788.

[6] A. È. Erëmenko and G. M. Levin, Periodic points of polynomials (Russian). Ukrain. Mat. Zh. **41** (1989), 1467–1471, 1581; translation in Ukrainian Math. J. **41** (1989), 1258–1262 (1990).

[7] Matts Essén and Shengjian Wu, Fix-points and a normal family of analytic functions. Complex Variables Theory Appl. **37** (1998), 171–178.

[8] Matts Essén and Shengjian Wu, Repulsive fixpoints of analytic functions with applications to complex dynamics. J. London Math. Soc. (2) **62** (2000), 139–149.

[9] Xuecheng Pang, Shared values and normal families. Analysis **22** (2002), 175–182.

[10] Xuecheng Pang and Lawrence Zalcman, Normal families and shared values. Bull. London Math. Soc. **32** (2000), 325-331.

[11] Heike Siebert, Fixed points and normal families of quasiregular mappings. J. Anal. Math. **98** (2006), 145–168.

[12] Shu Gui Wang and Sheng Jian Wu, Fixpoints of meromorphic functions and quasinormal families (Chinese). Acta Math. Sinica (Chin. Ser.) **45** (2002), 545–550.

[13] Yan Xu, On Montel's theorem and Yang's problem. J. Math. Anal. Appl. **305** (2005), 743–751.

[14] Lo Yang, Some recent results and problems in the theory of value-distribution. Proceedings of the symposium on value distribution theory in several complex variables, 157–171, Notre Dame Math. Lectures **12**, Univ. Notre Dame Press, Notre Dame, IN, 1992.

[15] Lawrence Zalcman, A heuristic principle in complex function theory. Amer. Math. Monthly **82** (1975), 813–817.

[16] Lawrence Zalcman, Normal families: new perspectives. Bull. Amer. Math. Soc. (N.S.) **35** (1998), 215–230.

Mathematisches Seminar der Christian–Albrechts–Universität zu Kiel, Ludewig–Meyn–Strasse 4, D–24098 Kiel, Germany

E-mail address: bergweiler@math.uni-kiel.de