# ENTIRE FUNCTIONS WITH JULIA SETS OF POSITIVE MEASURE

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Abstract. Let $f$ be a transcendental entire function for which the set of critical and asymptotic values is bounded. The Denjoy-Carleman-Ahlfors theorem implies that if the set of all $z$ for which $|f(z)| > R$ has $N$ components for some $R > 0$, then the order of $f$ is at least $N/2$. More precisely, we have $\log\log M(r, f) \geq \frac{1}{2}N \log r - O(1)$, where $M(r, f)$ denotes the maximum modulus of $f$. We show that if $f$ does not grow much faster than this, then the escaping set and the Julia set of $f$ have positive Lebesgue measure. However, as soon as the order of $f$ exceeds $N/2$, this need not be true. The proof requires a sharpened form of an estimate of Carleman and Tsuji related to the Denjoy-Carleman-Ahlfors theorem.

## 1. Introduction and results

The Julia set $J(f)$ of an entire function is defined as the set of all points in $\mathbb{C}$ where the iterates $f^n$ of $f$ do not form a normal family; see [3] for an introduction to transcendental dynamics.

McMullen [20] proved that $J(\sin(\alpha z + \beta))$ has positive Lebesgue measure and that $J(\lambda e^z)$ has Hausdorff dimension 2, for $\alpha, \beta, \lambda \in \mathbb{C}$, $\alpha, \lambda \neq 0$. The result on the Hausdorff dimension of $J(\lambda e^z)$ has been extended to large classes of functions; see [1, 6, 7, 27, 32]. It is the purpose of this paper to exhibit a class of functions whose Julia sets have positive measure. However, we begin by briefly describing the results on Hausdorff dimension.

We first recall that the Eremenko-Lyubich class $B$ consists of all transcendental entire functions for which the set of finite asymptotic values and critical values is bounded. Eremenko and Lyubich [13, Theorem 1] proved that if $f \in B$, then the escaping set $I(f)$ consisting of all points $z \in \mathbb{C}$ for which $f^n(z) \to \infty$ is contained in $J(f)$. In fact, it follows that $J(f) = \overline{I(f)}$ for $f \in B$. It is easily seen that $\sin(\alpha z + \beta) \in B$ and $\lambda e^z \in B$. McMullen actually proved that $I(\sin(\alpha z + \beta))$ has positive measure and $I(\lambda e^z)$ has Hausdorff dimension 2.

Next we note that the order $\varrho(f)$ of an entire function $f$ is defined by

$$\varrho(f) = \limsup_{r\to\infty} \frac{\log\log M(r, f)}{\log r}$$

where

$$M(r, f) = \max_{|z|=r} |f(z)|.$$

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Thus $\varrho(f)$ is the infimum of the set of all $\mu$ such that $|f(z)| \leq \exp(|z|^\mu)$ for large $|z|$, with $\varrho(f) = \infty$ if no such $\mu$ exists. We note that $\varrho(\lambda e^z) = \varrho(\sin(\alpha z + \beta)) = 1$.

McMullen's result on the Hausdorff dimension of $J(\lambda e^z)$ was substantially generalized by Barański [1] and, independently, Schubert [27]. They proved that if $f \in B$ and $\varrho(f) < \infty$, then $J(f)$ has Hausdorff dimension 2. The special case where $f$ has the form

$$f(z) = \int_0^z P(t)e^{Q(t)}dt + c, \tag{1.1}$$

with polynomials $P$ and $Q$ and with $c \in \mathbb{C}$, had been treated before by Taniguchi [32]. These functions are in $B$ and we have $\varrho(f) = \deg P$.

A generalisation of the result of Barański and Schubert was given in [7] where it is shown that if $f \in B$ and

$$q = \limsup_{r\to\infty} \frac{\log\log\log M(r,f)}{\log\log r} < \infty,$$

then $I(f)$ and hence $J(f)$ have Hausdorff dimension at least $(q+1)/q$. For further results on the Hausdorff dimension of Julia sets of entire functions we refer to [2, 8] and, in particular, the survey [31].

While McMullen's result on the Hausdorff dimension of $J(\lambda e^z)$ thus has prompted a lot of further research, there seem to be no papers whose main intention is to extend McMullen's result that $J(\sin(\alpha z+\beta))$ has positive measure to more general classes of functions. However, there are some papers devoted to ergodic properties of transcendental entire and meromorphic functions and their results in particular imply that the Julia sets of certain functions have positive measure.

We mention the work of Bock [9] whose results yield that if $f$ is a meromorphic function whose set of finite critical and asymptotic values is bounded and if there exist $\alpha > 0$ and $R > 0$ such that the set of all $z$ for which $|z| > R$ and $|f(z)| < e^t$ is contained in finitely many domains of the form $\{z : |\arg z - s| \leq t/(\log|z|)^\alpha\}$ for all large $t$, then $I(f)$ has positive measure. For example, this result applies to $f(z) = R(e^z)$, where $R$ is a rational function with $R(0) = R(\infty) = 0$, or to $f(z) = \sin P(z)$, where $P$ is a polynomial. Skorulski [28] considered functions of the form

$$f(z) = \frac{a\exp(z^p) + b\exp(-z^p)}{c\exp(z^p) + d\exp(-z^p)},$$

where $p \in \mathbb{N}$ and $a, b, c, d \in \mathbb{C}$, and Hemke [17] studied a class which contains all functions of the form (1.1). Both Skorulski and Hemke proved that $J(f)$ has positive measure for the functions considered, if the singularities of the inverse have a certain behavior under iteration. For a more detailed description of the above and other results on the measure of Julia and escaping sets we refer to the survey by Kotus und Urbański [18, Section 7].

We shall exhibit a condition which depends only on the growth of $f$ and which, for $f \in B$, implies that $I(f)$ and $J(f)$ have positive measure. Before stating this condition we recall that (one version of) the Denjoy-Carleman-Ahlfors-Theorem (see [14, p. 173], [16, Section 8.3] or [21, p. 309]) says that if $f$ is entire, $R > 0$ and $N$ denotes the number of components of

$$A_R = \{z \in \mathbb{C} : |f(z)| > R\},$$

then $N \leq \max\{1, 2\varrho(f)\}$. As we shall see below, we have $\varrho(f) \geq \frac{1}{2}$ for $f \in B$. (This seems to have been observed first in [4, 19]; see also [24, Lemma 3.5].) Thus

$N \leq 2\varrho(f)$ in this case. More precisely, we even have (see [16, Theorem 8.9] or [21, p. 312])

$$\log\log M(r,f) \geq \frac{N}{2}\log r - O(1) \tag{1.2}$$

as $r \to \infty$. We shall show that if $f \in B$ does not grow much faster than guaranteed by (1.2), then $I(f)$ and $J(f)$ have positive measure.

**Theorem 1.1.** *Let $f \in B$ and suppose that $A_R$ has $N$ components for some $R > 0$. If there exists $m \in \mathbb{N}$ such that*

$$\log\log M(r,f) \leq \left(\frac{N}{2} + \frac{1}{\log^m r}\right)\log r \tag{1.3}$$

*for large $r$, then $I(f)$ and $J(f)$ have positive Lebesgue measure. Here $\log^m$ denotes the $m$-th iterate of the logarithm.*

Theorem 1.1 will be a corollary of a more general result, which on the one hand works for a larger class of functions and which on the other hand has a condition more precise than (1.3). In order to state this result, we need some further notation.

For the functions we consider the behavior outside $A_R$ will be irrelevant. As in [8] we thus introduce tracts which are defined as follows. Let $U$ be an unbounded domain in $\mathbb{C}$ whose boundary consists of piecewise smooth curves. Suppose that the complement of $U$ is unbounded. Let $f$ be a complex-valued function whose domain of definition contains the closure $\overline{U}$ of $U$. Then $U$ is called a *direct tract* of $f$ if $f$ is holomorphic in $U$ and continuous in $\overline{U}$ and if there exists $R > 0$ such that $|f(z)| = R$ for $z \in \partial U$ while $|f(z)| > R$ for $z \in U$. If, in addition, the restriction $f : U \to \{z \in \mathbb{C} : |z| > R\}$ is a universal covering, then $U$ is called a *logarithmic tract* of $f$.

It is easily seen that if $f$ is a transcendental entire function and $R > 0$, then every component of $A_R$ is a direct tract. If all critical and asymptotic values of $f$ have modulus less than $R$, then every component of $A_R$ is a logarithmic tract. The Eremenko-Lyubich class consists of those functions for which such a value of $R$ exists.

For a direct tract $U$ of $f$ and $r > \min_{z\in U}|z|$ we define

$$M_U(r,f) = \max_{|z|=r,\, z\in U}|f(z)|.$$

To formulate a condition more precise than (1.3) we fix $\beta \in (0, 1/e)$ and note that the function $E_\beta(x) = e^{\beta x}$ has a repelling fixed point $\xi > e$ with multiplier

$$\mu = E_\beta'(\xi) = \beta E_\beta(\xi) = \beta\xi > 1.$$

Now Schröder's functional equation

$$\Phi(E_\beta(z)) = \mu\Phi(z) \tag{1.4}$$

has a unique solution $\Phi$ holomorphic in a neighborhood of $\xi$ and satisfying $\Phi(\xi) = 0$ and $\Phi'(\xi) = 1$. It is not difficult to see that $\Phi$ is real on the real axis and that $\Phi$ has a continuation $\Phi : [\xi, \infty) \to [0, \infty)$ so that (1.4) is satisfied for $\xi \leq z \leq \infty$. Moreover, $\Phi$ is increasing on the interval $[\xi, \infty)$ and we have $\lim_{x\to\infty}\Phi(x) = \infty$ while

$$\lim_{x\to\infty}\frac{\Phi(x)}{\log^m x} = 0$$

for all $m \in \mathbb{N}$. Thus $\Phi$ tends to $\infty$, but slower than any iterate of the logarithm. Hence the function

$$\varepsilon : (\xi, \infty) \to (0, \infty), \quad \varepsilon(x) = \frac{1}{\Phi(x)} \tag{1.5}$$

is decreasing and tends to 0 as $x \to \infty$, but it tends to 0 slower than any of the functions $1/\log^m x$. We mention that the function $\Phi$ also appears in recent work of Peter [22] on Hausdorff measure of exponential Julia sets.

**Theorem 1.2.** *Let $f$ be a function which has $N$ (pairwise disjoint) logarithmic tracts $U_1, U_2, \ldots, U_N$. Put*

$$U = \bigcup_{j=1}^{N} U_j, \quad M_U(r, f) = \max_{j=1,\ldots,N} M_{U_j}(r, f) = \max_{|z|=r,\, z \in U} |f(z)|$$

*and*

$$I_U(f) = \left\{ z \in U : f^n(z) \in U \text{ for all } n \in \mathbb{N}, \lim_{n\to\infty} f^n(z) = \infty \right\}.$$

*Let $\varepsilon(x)$ be defined by (1.5), for some fixed $\beta \in (0, 1/e)$, and suppose that*

$$\log \log M_U(r, f) \le \left( \frac{N}{2} + \varepsilon(r) \right) \log r$$

*for large $r$. Then $I_U(f)$ has positive Lebesgue measure.*

Since $I(f) \subset J(f)$ for $f \in B$ by the result of Eremenko and Lyubich [13, Theorem 1] already quoted, Theorem 1.1 follows indeed from Theorem 1.2.

In Section 4 we will give an example which shows that the function $\varepsilon(r)$ in Theorem 1.2 and the function $1/\log^m r$ in Theorem 1.1 cannot be replaced by a positive constant $\varepsilon$.

The proof of Theorem 1.2 will use some ideas connected to the Denjoy-Carleman-Ahlfors-Theorem. One way to prove the latter theorem is based on an estimate of Carleman [10]. We shall use it in the form given by Tsuji [33, p. 116]. To formulate this result, let $U$ be a direct tract of $f$ and put

$$\theta(r) = \operatorname{meas}\left( \left\{ t \in [0, 2\pi] : re^{it} \in U \right\} \right). \tag{1.6}$$

Choose $r_0 > 0$ such that $\{z \in \mathbb{C} : |z| = r\} \cap U \ne \emptyset$ and hence $\theta(r) > 0$ for $r \ge r_0$. For $r \ge r_0$, put $\theta^*(r) = \theta(r)$ if $\{z \in \mathbb{C} : |z| = r\} \not\subset U$ and put $\theta^*(r) = \infty$ and thus $1/\theta^*(r) = 0$ otherwise. Tsuji's result says that for $0 < \kappa < 1$ there exists a constant $C$ such that

$$\log \log M_U(r, f) \ge \pi \int_{r_0}^{\kappa r} \frac{dt}{t\theta^*(t)} - C \tag{1.7}$$

for $r > r_0/\kappa$.

For logarithmic tracts we have

$$\{z \in \mathbb{C} : |z| = r\} \not\subset U \tag{1.8}$$

and thus $\theta^*(r) = \theta(r) \le 2\pi$ for $r \ge r_0$. Hence (1.7) yields

$$\log \log M_U(r, f) \ge \frac{1}{2} \int_{r_0}^{\kappa r} \frac{dt}{t} - C = \frac{1}{2} \log r - C - \log \frac{\kappa}{r_0}$$

for large $r_0$ and $r > r_0/\kappa$. In particular, it follows that $\varrho(f) \ge \frac{1}{2}$ for $f \in B$, as mentioned above.

To prove Theorem 1.2 we shall need a refinement of (1.7) in the case that (1.8) holds for large $r$.

**Theorem 1.3.** *Let $f$ be a function with a direct tract $U$. Suppose that (1.8) holds for all large $r$ and that*

$$\pi \int_{r_0}^{\kappa r} \frac{dt}{t\theta(t)} \geq \lambda \log r - O(1) \tag{1.9}$$

*as $r \to \infty$, where $\lambda \geq \frac{1}{2}$. Let $0 < \beta < \lambda$ and put*

$$V = \left\{ z \in U : |f(z)| \geq \exp\left(|z|^\beta\right) \right\}$$

*and*

$$\psi(r) = \operatorname{meas}\left(\left\{ t \in [0, 2\pi] : re^{it} \in V \right\}\right). \tag{1.10}$$

*Then for $0 < \kappa < 1$ there exist constants $C$ and $r_0$ such*

$$\log\log M_U(r, f) \geq \pi \int_{r_0}^{\kappa r} \frac{dt}{t\psi(t)} - C$$

*for $r \geq r_0/\kappa$.*

Note that (1.8) implies that (1.9) is always satisfied for $\lambda = \frac{1}{2}$. Thus we can always take $\beta < \frac{1}{2}$. In particular, any $\beta$ in the interval $(0, 1/e)$ is admissible.

As mentioned, the hypothesis (1.8) always holds for logarithmic tracts. It clearly also holds if $f$ has several direct tracts. Possibly the hypothesis (1.8) can be omitted from Theorem 1.3 altogether.

We shall prove Theorem 1.3 in Section 2 and Theorem 1.2 in Section 3.

## 2. Proof of Theorem 1.3

In this section we prove Theorem 1.3 following Tsuji [33, Section III.17]; see also [14, Section 5.1] and, for a slightly different approach, [16, Section 8.1]. The original result works for subharmonic functions and the main difference here is that we shall apply Tsuji's reasoning to the function

$$v(z) = \log|f(z)| - |z|^\beta \tag{2.1}$$

which is not subharmonic. However, $|z|^\beta$ is small compared to $\log|f(z)|$ so that $v$ can be considered as a small perturbation of a subharmonic function. This makes Tsuji's arguments still work.

The following lemma [33, p. 112] is known as Wirtinger's inequality.

**Lemma 2.1.** *Let $f$ and $f'$ be continuous in $[a, b]$ and $f(a) = f(b) = 0$. Then*

$$\int_a^b f'(x)^2 dx \geq \frac{\pi^2}{(b-a)^2} \int_a^b f(x)^2 dx.$$

Define $R > 0$ such that $|f(z)| = R$ for $z \in \partial U$. Let $v$ be defined by (2.1) and let

$$V = \{z \in U : v(z) \geq 0\} \subset U.$$

Define $V_r = \{\theta \in [t_r, 2\pi + t_r] : re^{i\theta} \in V\}$ where $t_r$ is chosen such that $re^{it_r} \notin U$ and put

$$m(r) = m_v(r) = \frac{1}{2\pi} \int_{V_r} v(re^{it})^2 dt.$$

Hence $\text{meas}(V_r) = \psi(r)$, where $\psi(r)$ is defined by (1.10). Recall also that $\theta(r)$ is defined by (1.6).

Next we prove the following.

**Lemma 2.2.** *There exist $c, r_0 > 0$ such that $m(r) \geq cr^{2\lambda}$ for $r \geq r_0$.*

*Proof.* Without loss of generality we may assume that $R = 1$. Let $u(z) = \log |f(z)|$ for $z \in U$ and $u(z) = 0$ outside $U$. Then $u \geq 0$ and $u$ is subharmonic. Let

$$m_u(r) = \frac{1}{2\pi} \int_0^{2\pi} u(re^{i\theta})^2 d\theta.$$

Note that $m_u(r) \leq (\log M_U(r, f))^2$. Inequality (1.7) is actually a corollary of a more general result [16, Theorem 8.2] which says that

$$\log \sqrt{m_u(r)} \geq \pi \int_{r_0}^{\kappa r} \frac{dt}{t\theta(t)} - C,$$

for $r > r_0/\kappa$. (Note that the notation in [16] is different from ours. They are related as follows: $I^2(r) = 2\pi m_u(r)$ and $\alpha(r) \geq \pi/\theta(r)$; cf. [16, equations (8.1.8) and (8.1.10)].)

Using (1.9) we obtain $\log m_u(r) \geq 2\lambda \log r - O(1)$ and thus $m_u(r) \geq c' r^{2\lambda}$ for some $c' > 0$, if $r > r_0/\kappa$.

To obtain a similar estimate for $m(r)$, first write

$$\int_{V_r} v(re^{i\theta})^2 d\theta = \int_{V_r} u(re^{i\theta})^2 d\theta - 2 \int_{V_r} u(re^{i\theta}) r^\beta d\theta + \int_{V_r} r^{2\beta} d\theta.$$

By the Cauchy-Schwarz inequality,

$$\int_{V_r} u(re^{i\theta}) r^\beta d\theta \leq \sqrt{\int_{V_r} u(re^{i\theta})^2 d\theta} \sqrt{\int_{V_r} r^{2\beta} d\theta} \leq \sqrt{2\pi} r^\beta \sqrt{\int_{V_r} u(re^{i\theta})^2 d\theta}.$$

Hence

$$\begin{aligned} \int_{V_r} v(re^{i\theta})^2 d\theta &\geq \int_{V_r} u(re^{i\theta})^2 d\theta - \sqrt{8\pi} r^\beta \sqrt{\int_{V_r} u(re^{i\theta})^2 d\theta} \\ &= \sqrt{\int_{V_r} u(re^{i\theta})^2 d\theta} \left( \sqrt{\int_{V_r} u(re^{i\theta})^2 d\theta} - \sqrt{8\pi} r^\beta \right). \end{aligned} \tag{2.2}$$

We have

$$\begin{aligned} c' r^{2\lambda} &\leq m_u(r) \\ &= \frac{1}{2\pi} \int_{V_r} u(re^{i\theta})^2 d\theta + \frac{1}{2\pi} \int_{\{\theta:\, 0 \leq u(re^{i\theta}) < r^\beta\}} u(re^{i\theta})^2 d\theta \\ &\leq \frac{1}{2\pi} \int_{V_r} u(re^{i\theta})^2 d\theta + r^{2\beta}. \end{aligned}$$

Hence

$$\int_{V_r} u(re^{i\theta})^2 d\theta \geq 2\pi c' r^{2\lambda} - 2\pi r^{2\beta} \geq c'' r^{2\lambda}$$

for some $c'' > 0$ and $r \geq r_0$, provided $r_0$ is large enough.

Hence (2.2) yields

$$m(r) = \frac{1}{2\pi} \int_{V_r} v(re^{i\theta})^2 d\theta \geq \frac{1}{2\pi} \sqrt{c''} r^\lambda (\sqrt{c''} r^\lambda - \sqrt{8\pi} r^\beta) \geq cr^{2\lambda},$$

for some $c > 0$ and $r \geq r_0$, if $r_0$ is sufficiently large. □

Next we prove (still following Tsuji [33, pp. 113–114]) that $m(r)$ is a convex function of $\log r$ for large $r$.

**Lemma 2.3.** *The second derivative of $m(r)$ exists except possibly for a discrete set of $r$-values and there exist $r_0 > 0$ such that*

$$\frac{d^2 m(r)}{d(\log r)^2} \geq 0,$$

*for all $r \geq r_0$ for which the derivative exists. More precisely,*

$$\frac{d^2 m(r)}{d(\log r)^2} \geq \frac{1}{2m(r)} \left( \frac{dm(r)}{d \log r} \right)^2 + \frac{m(r)}{2} q(r) \tag{2.3}$$

*where*

$$q(r) = \left( \frac{2\pi}{\psi(r)} \right)^2 - \frac{4\beta^2}{\sqrt{c}} r^{\beta - \lambda} \geq 0 \tag{2.4}$$

*for $r \geq r_0$. The constant $c$ is the same as in Lemma 2.2.*

*Proof.* The Laplacian in polar coordinates is given by

$$\begin{aligned} \frac{1}{r^2} \left( \frac{\partial^2 v(re^{i\theta})}{\partial(\log r)^2} + \frac{\partial^2 v(re^{i\theta})}{\partial \theta^2} \right) &= \Delta v(re^{i\theta}) \\ &= \Delta \left( \log |f(re^{i\theta})| - r^\beta \right) \\ &= -\beta^2 r^{\beta - 2}. \end{aligned} \tag{2.5}$$

Since $v$ is real analytic except at the zeros of $f$, the set $V_r$ consists of finitely many intervals $[\alpha_j(r), \beta_j(r)]$. Moreover, the set of points where $v = 0$ and $\partial v / \partial \theta = 0$ is discrete. By the Implicit Function Theorem, there exists a discrete set of $r$-values such that in the complementary intervals of this set the number of intervals $[\alpha_j(r), \beta_j(r)]$ is constant and the endpoints $\alpha_j(r)$ and $\beta_j(r)$ are differentiable functions of $r$. Since $v(re^{i\alpha_j(r)}) = v(re^{i\beta_j(r)}) = 0$ for all $j$ we obtain

$$\begin{aligned} &\frac{dm(r)}{d \log r} \\ &= \frac{1}{2\pi} \sum_j \frac{d}{d \log r} \int_{\alpha_j(r)}^{\beta_j(r)} v(re^{i\theta})^2 d\theta \\ &= \frac{1}{2\pi} \sum_j \int_{\alpha_j(r)}^{\beta_j(r)} \frac{\partial v(re^{i\theta})^2}{\partial \log r} d\theta + v(re^{i\beta_j(r)})^2 \frac{d\beta_j(r)}{d \log r} - v(re^{i\alpha_j(r)})^2 \frac{d\alpha_j(r)}{d \log r} \\ &= \frac{1}{2\pi} \sum_j \int_{\alpha_j(r)}^{\beta_j(r)} \frac{\partial v(re^{i\theta})^2}{\partial \log r} d\theta \\ &= \frac{1}{\pi} \int_{V_r} v(re^{i\theta}) \frac{\partial v(re^{i\theta})}{\partial \log r} d\theta, \end{aligned}$$

except for a discrete set of $r$-values.

By the same reasoning,

$$\frac{d^2 m(r)}{d(\log r)^2} = \frac{1}{\pi} \int_{V_r} \left( \left( \frac{\partial v(re^{i\theta})}{\partial \log r} \right)^2 + v(re^{i\theta}) \frac{\partial^2 v(re^{i\theta})}{\partial(\log r)^2} \right) d\theta. \tag{2.6}$$

Now

$$\frac{\partial^2}{\partial\theta^2}\left(v(re^{i\theta})^2\right) = \frac{\partial}{\partial\theta}\left(2v(re^{i\theta})\frac{\partial v(re^{i\theta})}{\partial\theta}\right)$$
$$= 2v(re^{i\theta})\frac{\partial^2 v(re^{i\theta})}{\partial\theta^2} + 2\left(\frac{\partial v(re^{i\theta})}{\partial\theta}\right)^2,$$

and since $v(re^{i\alpha_j(r)}) = v(re^{i\beta_j(r)}) = 0$ we have

$$\int_{\alpha_j(r)}^{\beta_j(r)} \frac{\partial^2}{\partial\theta^2}\left(v(re^{i\theta})^2\right) d\theta = \left[2v(re^{i\theta})\frac{\partial v(re^{i\theta})}{\partial\theta}\right]_{\alpha_j(r)}^{\beta_j(r)} = 0$$

for all $j$. Thus

$$\int_{V_r} v(re^{i\theta})\frac{\partial^2 v(re^{i\theta})}{\partial\theta^2} d\theta = -\int_{V_r}\left(\frac{\partial v(re^{i\theta})}{\partial\theta}\right)^2 d\theta$$

and using (2.5) and (2.6) we obtain

$$\frac{d^2 m(r)}{d(\log r)^2} = \frac{1}{\pi}\int_{V_r}\left(\left(\frac{\partial v(re^{i\theta})}{\partial \log r}\right)^2 + \left(\frac{\partial v(re^{i\theta})}{\partial\theta}\right)^2 - v(re^{i\theta})\beta^2 r^\beta\right) d\theta.$$

Let us write

$$J_1 = \frac{1}{\pi}\int_{V_r}\left(\frac{\partial v(re^{i\theta})}{\partial \log r}\right)^2 d\theta,$$
$$J_2 = \frac{1}{\pi}\int_{V_r}\left(\frac{\partial v(re^{i\theta})}{\partial\theta}\right)^2 d\theta,$$
$$J_3 = \frac{1}{\pi}\int_{V_r} v(re^{i\theta})\beta^2 r^\beta d\theta.$$

To estimate $J_1$ we use the Cauchy-Schwarz inequality to obtain

$$\left(\frac{dm(r)}{d\log r}\right)^2 = \left(\frac{1}{\pi}\int_{V_r} v(re^{i\theta})\frac{\partial v(re^{i\theta})}{\partial \log r} d\theta\right)^2$$
$$\leq \frac{1}{\pi^2}\int_{V_r} v(re^{i\theta})^2 d\theta \int_{V_r}\left(\frac{\partial v(re^{i\theta})}{\partial \log r}\right)^2 d\theta$$
$$\leq 2m(r)J_1.$$

Hence

$$J_1 \geq \frac{1}{2m(r)}\left(\frac{dm(r)}{d\log r}\right)^2.$$

Recall that $V_r$ is a union of intervals $[\alpha_j(r), \beta_j(r)]$ so that

$$\psi(r) = \sum_j \left(\beta_j(r) - \alpha_j(r)\right).$$

Using Wirtinger's inequality on each of these intervals we get

$$\frac{1}{\pi}\int_{\alpha_j(r)}^{\beta_j(r)}\left(\frac{\partial v(re^{i\theta})}{\partial\theta}\right)^2 d\theta \geq \frac{\pi}{(\beta_j(r) - \alpha_j(r))^2}\int_{\alpha_j(r)}^{\beta_j(r)} v(re^{i\theta})^2 d\theta$$
$$\geq \frac{\pi}{\psi(r)^2}\int_{\alpha_j(r)}^{\beta_j(r)} v(re^{i\theta})^2 d\theta.$$

Summing over all $j$ yields

$$
\begin{aligned}
J_2 &= \frac{1}{\pi} \sum_j \int_{\alpha_j(r)}^{\beta_j(r)} \left( \frac{\partial v(re^{i\theta})}{\partial \theta} \right)^2 d\theta \\
&\geq \frac{\pi}{\psi(r)^2} \sum_j \int_{\alpha_j(r)}^{\beta_j(r)} v(re^{i\theta})^2 d\theta \\
&= \frac{\pi}{\psi(r)^2} \int_{V_r} v(re^{i\theta})^2 d\theta \\
&= \frac{2\pi^2}{\psi(r)^2} m(r).
\end{aligned}
$$

To estimate $J_3$ we use the Cauchy-Schwarz inequality again to obtain

$$
\begin{aligned}
J_3 &= \frac{\beta^2 r^\beta}{\pi} \int_{V_r} v(re^{i\theta}) d\theta \\
&\leq \frac{\beta^2 r^\beta}{\pi} \left( \int_{V_r} v(re^{i\theta})^2 d\theta \right)^{1/2} \left( \int_{V_r} 1\, d\theta \right)^{1/2} \\
&\leq 2\beta^2 r^\beta \sqrt{m(r)} \\
&= 2\beta^2 r^{\beta-\lambda} r^\lambda \sqrt{m(r)}.
\end{aligned}
$$

Using Lemma 2.2 we obtain

$$
J_3 \leq \frac{2\beta^2}{\sqrt{c}} r^{\beta-\lambda} m(r).
$$

Hence

$$
\begin{aligned}
\frac{d^2 m(r)}{d(\log r)^2} &\geq J_1 + J_2 - J_3 \\
&\geq \frac{1}{2m(r)} \left( \frac{dm(r)}{d\log r} \right)^2 + \frac{2\pi^2}{\psi(r)^2} m(r) - \frac{2\beta^2}{\sqrt{c}} r^{\beta-\lambda} m(r) \\
&= \frac{1}{2m(r)} \left( \frac{dm(r)}{d\log r} \right)^2 + \frac{m(r)}{2} \left( \left( \frac{2\pi}{\psi(r)} \right)^2 - \frac{4\beta^2}{\sqrt{c}} r^{\beta-\lambda} \right).
\end{aligned}
$$

This yields (2.3). Since $\psi(r) \leq 2\pi$ and $\beta < \lambda$ there exists $r_0 > 0$ such that (2.4) holds for $r \geq r_0$. This completes the proof of Lemma 2.3. □

By (2.4) we may define $\alpha$ and $\tilde{\alpha}$ by $\tilde{\alpha}(r) = \alpha(\log r) = \sqrt{q(r)}$ for $r \geq r_0$. Put $\mu(t) = m(e^t)$. We use the change of variables $r = e^t$, so $\alpha(t) = \tilde{\alpha}(r)$ and $\mu(t) = m(r)$. Now inequality (2.3) becomes

$$
\mu''(t) \geq \frac{\mu'(t)^2}{2\mu(t)} + \frac{1}{2}\alpha(t)^2 \mu(t).
$$

With $\alpha(t)$ replaced by $2\pi/\theta^*(t)$ and $\mu(t) = m(e^t)$ replaced by $m_u(e^t)$ this equation was obtained by Tsuji [33, p. 114, equation (11)]. We continue to follow [33] closely and obtain

$$
\left( \frac{\mu''(t)}{\mu'(t)} \right)^2 \geq \alpha(t)^2 \tag{2.7}
$$

as there.

We now argue that in fact also $\mu'(t) \geq 0$ for large enough $t$. Lemma 2.2 implies that $\mu(t) = m(e^t) \geq ce^{2\lambda t}$ for all $t \geq \log r_0$. This means that $\mu'(t) = rm'(r) > 0$ for some $t$ because otherwise $\mu$ would be bounded. Since also $\mu''(t) = d^2m(r)/d(\log r)^2 \geq 0$ for large $t$ this implies that actually $\mu'(t) > 0$ for all large $t$, say $t \geq \log r_0$.

Hence from (2.7) we get

$$\frac{\mu''(t)}{\mu'(t)} \geq \alpha(t) \quad \text{for all } t \geq \log r_0.$$

To conclude the proof of Theorem 1.3, let $\tau > t_0 = \log r_0$ and note that

$$\log \mu'(\tau) - \log \mu'(t_0) = \int_{t_0}^{\tau} \frac{\mu''(\rho)}{\mu'(\rho)} d\rho \geq \int_{t_0}^{\tau} \alpha(\rho) d\rho,$$

so

$$\mu'(\tau) \geq \mu'(t_0) \exp\left\{ \int_{t_0}^{\tau} \alpha(\rho) d\rho \right\}.$$

With $t = \log r > t_0$ we have, since $\mu(t)$ is increasing for $t \geq \log r_0$,

$$\mu(t) \geq \mu(t) - \mu(t_0) = \int_{t_0}^{t} \mu'(\tau) d\tau \geq \mu'(t_0) \int_{t_0}^{t} \exp\left\{ \int_{t_0}^{\tau} \alpha(\rho) d\rho \right\} d\tau. \tag{2.8}$$

With $\rho = \log s$ and $\tau = \log \sigma$ we get

$$\mu(t) \geq \mu'(t_0) \int_{r_0}^{r} \exp\left\{ \int_{r_0}^{\sigma} \frac{\tilde{\alpha}(s)}{s} ds \right\} \frac{d\sigma}{\sigma}.$$

For $r \geq r_0/\kappa$, with $0 < \kappa < 1$, we thus have, using $\log(1/\kappa) \geq 1 - \kappa$,

$$\begin{aligned} \mu(t) &\geq \mu'(t_0) \int_{\kappa r}^{r} \exp\left\{ \int_{r_0}^{\sigma} \frac{\tilde{\alpha}(s)}{s} ds \right\} \frac{d\sigma}{\sigma} \\ &\geq \mu'(t_0)(1-\kappa) \exp\left\{ \int_{r_0}^{\kappa r} \frac{\tilde{\alpha}(s)}{s} ds \right\}. \end{aligned}$$

With $c_0 = \mu'(t_0)$ thus

$$\mu(t) \geq c_0(1-\kappa) \exp\left\{ \int_{r_0}^{\kappa r} \frac{\tilde{\alpha}(s)}{s} ds \right\}. \tag{2.9}$$

We want to estimate the integral on the right side. Using $\sqrt{x} \geq x$ for $x \in [0,1]$ we obtain

$$\begin{aligned} \tilde{\alpha}(s) &= \frac{2\pi}{\psi(s)} \sqrt{1 - \frac{\beta^2 \psi(s)^2}{\pi^2 \sqrt{c}} s^{\beta-\lambda}} \\ &\geq \frac{2\pi}{\psi(s)} \left( 1 - \frac{\beta^2 \psi(s)^2}{\pi^2 \sqrt{c}} s^{\beta-\lambda} \right) \\ &= \frac{2\pi}{\psi(s)} - \gamma \psi(s) s^{\beta-\lambda}, \end{aligned}$$

for $s \geq r_0$, where $\gamma = 2\beta^2/(\pi\sqrt{c})$. Therefore,

$$\int_{r_0}^{\kappa r} \frac{\tilde{\alpha}(s)}{s} ds \geq 2\pi \int_{r_0}^{\kappa r} \frac{ds}{s\psi(s)} - \gamma \int_{r_0}^{\kappa r} \psi(s) s^{\beta-\lambda-1} ds.$$

But, since $\beta < \lambda$ and $\psi(s) \le 2\pi$,

$$\gamma \int_{r_0}^{\kappa r} \psi(s) s^{\beta-\lambda-1} ds \le c_1$$

for some constant $c_1 > 0$. Hence (2.9) yields

$$\mu(t) \ge c_0(1-\kappa)e^{-c_1} \exp\left\{2\pi \int_{r_0}^{\kappa r} \frac{ds}{s\psi(s)} ds\right\}.$$

Recalling that $t = \log r$ and $m(r) = \mu(t)$, we get

$$m(r) \ge c_2 \exp\left\{2\pi \int_{r_0}^{\kappa r} \frac{ds}{s\psi(s)} ds\right\},$$

where $c_2 = c_0(1-\kappa)e^{c_1}$.

From this and the fact that

$$\log\log M_U(r,f) \ge \log \max_{|z|=r} v(z) \ge \log\sqrt{m(r)} = \frac{1}{2}\log m(r),$$

Theorem 1.3 follows.

*Remark* 2.1. With some more effort (see again [33]), one can show that

$$m(\rho) \le \frac{2e^{c_1+1}m(r)}{1-\kappa} \exp\left\{-2\pi \int_{\rho}^{\kappa r} \frac{\tilde{\alpha}(s)}{s} ds\right\}$$

for $r_0 \le \rho < \kappa r$. Using this it follows that the constant $C$ in Theorem 1.2 only depends on $\kappa$, $r_0$ and $\beta$.

## 3. Proof of Theorem 1.2

We begin by describing the *logarithmic change of variable* which was the main tool used by Eremenko and Lyubich [13, Section 2] to study the dynamics of a function $f \in B$.

We may assume that $|f(z)| = R$ for $z \in \partial U = \bigcup_{j=1}^{N} \partial U_j$ and, increasing $R$ if necessary, we may also achieve that $0 \notin U$. Put $H = \{z \in \mathbb{C} : \operatorname{Re} z > \log R\}$ and fix $j \in \{1, \dots, N\}$. Since $f : U_j \to \{z \in \mathbb{C} : |z| > R\}$ and $\exp : H \to \{z \in \mathbb{C} : |z| > R\}$ are universal coverings there exists a biholomorphic map $G : U_j \to H$ such that $f|_{U_j} = \exp \circ G$. Doing this for each $j$ we obtain a map $G : U \to H$ and putting $W = \exp^{-1}(U)$ we can thus define $F : W \to H$, $F(z) = G(e^z)$. Thus $\exp F(z) = f(e^z)$ and $F$ maps every component of $W$ univalently onto $H$. We say that $F$ is the function obtained from $f$ by a logarithmic change of variable.

As noted by Eremenko and Lyubich, this logarithmic change of variable had already been used by Teichmüller in value distribution theory. In fact, the biholomorphic map $G : U_j \to H$ appears already in the work of Speiser [29, p. 295], and probably even earlier.

If $\phi$ is a branch of the inverse function of $F$ and if $w \in H$, then $\phi$ is defined in particular in the disk of radius $\operatorname{Re} w - \log R$ around $w$. Thus Koebe's one quarter theorem implies that $\phi(H)$ contains a disk of radius $\frac{1}{4}|\phi'(w)|(\operatorname{Re} w - \log R)$ around $\phi(w)$. Since $W$ and hence $\phi(H)$ do not contain vertical segments of length greater than $2\pi$, and thus in particular no disc of radius greater than $\pi$, it follows that

$$|\phi'(w)| \le \frac{4\pi}{\operatorname{Re} w - \log R}. \tag{3.1}$$

In terms of $F$ this inequality takes the form [13, Lemma 1]

$$|F'(z)| \geq \frac{\operatorname{Re} F(z) - \log R}{4\pi} \tag{3.2}$$

for $z \in W$.

Another tool we shall use is the Besicovitch covering lemma [11, Theorem 3.2.1]. Here we denote the ball of radius $r$ around a point $x \in \mathbb{R}^n$ by $B(x, r)$.

**Lemma 3.1.** *Let $K \subset \mathbb{R}^n$ be bounded and $r : K \to (0, \infty)$. Then there exists an at most countable subset $L$ of $K$ satisfying*

$$K \subset \bigcup_{x \in L} B(x, r(x))$$

*such that no point in $\mathbb{R}^n$ is contained in more than $4^{2n}$ of the balls $B(x, r(x))$, $x \in L$.*

We now begin with the actual proof of Theorem 1.2. We may assume that $R$ is so large that $E_\beta^n(x) \to \infty$ as $n \to \infty$ for $x > \log R$, with $E_\beta(x) = e^{\beta x}$ as in the introduction. For $j = 1, \dots, N$ we put

$$V_j = \left\{ z \in U_j : |f(z)| \geq \exp\left(|z|^\beta\right) \right\}$$

and denote by $\psi_j(r)$ the measure of the set of all $t \in [0, 2\pi]$ such that $re^{it} \in V_j$. It follows from Theorem 1.3 that for $0 < \kappa < 1$ there exist constants $r_0$ and $C$ such that

$$\log \log M_U(r, f) \geq \pi \int_{r_0}^{\kappa r} \frac{dt}{t \psi_j(t)} - C$$

for $r > r_0/\kappa$. Hence

$$\log \log M_U(r, f) \geq \pi \int_{r_0}^{\kappa r} \left( \frac{1}{N} \sum_{j=1}^{N} \frac{1}{\psi_j(t)} \right) \frac{dt}{t} - C.$$

By the Cauchy-Schwarz inequality we have

$$N^2 = \left( \sum_{j=1}^{N} \frac{\sqrt{\psi_j(t)}}{\sqrt{\psi_j(t)}} \right)^2 \leq \left( \sum_{j=1}^{N} \frac{1}{\psi_j(t)} \right) \cdot \left( \sum_{j=1}^{N} \psi_j(t) \right).$$

With

$$\psi(t) = \sum_{j=1}^{N} \psi_j(t)$$

we deduce that

$$\sum_{j=1}^{N} \frac{1}{\psi_j(t)} \geq \frac{N^2}{\psi(t)}$$

and hence that

$$\log \log M_U(r, f) \geq N\pi \int_{r_0}^{\kappa r} \frac{1}{\psi(t)} \frac{dt}{t} - C.$$

By hypothesis we have

$$\begin{aligned}\log\log M_U(r,f) &\leq \frac{N}{2}\log r + \varepsilon(r)\log r\\ &= \frac{N}{2}\left(\int_{r_0}^{\kappa r}\frac{dt}{t} + \log\frac{r_0}{\kappa}\right) + \varepsilon(r)\log r.\end{aligned}$$

It follows from the last two inequalities that

$$\begin{aligned}\varepsilon(r)\log r + C + \frac{N}{2}\log\frac{r_0}{\kappa} &\geq N\pi\int_{r_0}^{\kappa r}\left(\frac{1}{\psi(t)} - \frac{1}{2\pi}\right)\frac{dt}{t}\\ &= N\pi\int_{r_0}^{\kappa r}\frac{2\pi-\psi(t)}{2\pi\psi(t)}\frac{dt}{t}\\ &\geq \frac{N}{4\pi}\int_{r_0}^{\kappa r}(2\pi-\psi(t))\frac{dt}{t}.\end{aligned}$$

Since $\varepsilon(r)$ is decreasing and $\varepsilon(r)\log r \to \infty$ as $r \to \infty$ we obtain

$$\begin{aligned}\int_{r_0}^{r}(2\pi-\psi(t))\frac{dt}{t} &\leq \frac{4\pi}{N}\left(\varepsilon\left(\frac{r}{\kappa}\right)\log\frac{r}{\kappa} + C + \frac{N}{2}\log\frac{r_0}{\kappa}\right)\\ &\leq \frac{5\pi}{N}\varepsilon(r)\log r\end{aligned} \tag{3.3}$$

for large $r$.

Let now $F$ be the function obtained from $f$ by the logarithmic change of variable. With $W = \bigcup_{j=1}^{N}\exp^{-1}(U_j)$ and $H = \{z \in \mathbb{C} : \operatorname{Re} z > \log R\}$ we thus have

$$F : W \to H,\ F(z) = \log f(e^z),$$

for some branch of the logarithm. Moreover, $F$ maps every component of $W$ bijectively onto $H$.

The real part of $F$ is large on the set $L = \bigcup_{j=1}^{N}\exp^{-1}(V_j)$. In fact,

$$L = \{z \in W : \operatorname{Re} F(z) \geq \exp(\beta \operatorname{Re} z)\}.$$

We put

$$T = \{z \in L : F^n(z) \in L \text{ for all } n \in \mathbb{N}\}.$$

For $z \in T$ we then have

$$\operatorname{Re} F^n(z) \geq E_\beta^n(\operatorname{Re} z)$$

and thus Re $F^n(z) \to \infty$ as $n \to \infty$. It follows that

$$|f^n(e^z)| = \exp(\operatorname{Re} F^n(z)) \to \infty$$

so that $\exp(T) \subset I_U(f)$.

We shall show that $\operatorname{area}(T) > 0$. This implies that $\operatorname{area}(I_U(f)) > 0$. In order to prove that $\operatorname{area}(T) > 0$ we consider for $n \geq 0$ the set

$$T_n = \{z \in L : F^k(z) \in L \text{ for } 0 \leq k \leq n\}$$

so that $T_0 = L$. Then

$$T = \bigcap_{n=1}^{\infty} T_n.$$

Let $S = \mathbb{C}\backslash L$ and put

$$\Psi(x) = \operatorname{meas}\{y \in [0,2\pi] : x + iy \in S\}$$

for $x > \log R$. Since for $x + iy \in L$ we have $e^x e^{iy} \in \bigcup_{j=1}^N V_j$ it follows that

$$\Psi(x) = 2\pi - \psi(e^x).$$

From (3.3) we deduce that

$$\int_{\log r_0}^{x} \Psi(s)ds = \int_{r_0}^{e^x} \Psi(\log t)\frac{dt}{t} = \int_{r_0}^{e^x} (2\pi - \psi(t))\frac{dt}{t} \leq \frac{5\pi}{N}\varepsilon(e^x)x.$$

We put $\delta(x) = \varepsilon(e^x)$. It follows that if $x_0 \geq \log r_0$, then

$$\int_{x_0}^{x} \Psi(s)ds \leq \frac{5\pi}{N}\delta(x)x.$$

For $z = x + iy \in \mathbb{C}$ with $x > 2\log R$ we denote by $Q(z)$ the square of sidelength $x$ centered at $z$. Thus

$$Q(z) = \left\{\zeta \in \mathbb{C} : |\mathrm{Re}\,\zeta - x| \leq \frac{1}{2}x, |\,\mathrm{Im}\,\zeta - y| \leq \frac{1}{2}x\right\}.$$

Now

$$\mathrm{area}\left(\{z \in S : \{\xi_1 \leq \mathrm{Re}\,z \leq \xi_2, y_0 \leq \mathrm{Im}\,z \leq y_0 + 2\pi\}\right) = \int_{\xi_1}^{\xi_2} \Psi(s)ds$$

for $\log R < \xi_1 < \xi_2$ and $y_0 \in \mathbb{R}$. Since $Q(z)$ can be covered by $[\frac{x}{2\pi} + 1]$ horizontal strips of width $2\pi$ we obtain

$$\begin{aligned} \mathrm{area}(Q(z) \cap S) &\leq \left(\frac{x}{2\pi} + 1\right) \int_{\frac{1}{2}x}^{\frac{3}{2}x} \Psi(s)ds \\ &\leq \left(\frac{x}{2\pi} + 1\right) \frac{5\pi}{N}\delta\left(\frac{3}{2}x\right)\frac{3}{2}x \\ &\leq \frac{4}{N}\delta(x)x^2 \end{aligned} \tag{3.4}$$

for large $x$. Recall that for measurable sets $A, B \subset \mathbb{C}$ the density of $A$ in $B$ is defined by

$$\mathrm{dens}(A, B) = \frac{\mathrm{area}(A \cap B)}{\mathrm{area}(B)}.$$

With this notation (3.4) takes the form

$$\mathrm{dens}(S, Q(z)) \leq \frac{4}{N}\delta(x). \tag{3.5}$$

We now fix $n \in \mathbb{N}$ and consider $u \in T_{n-1}\backslash T_n$ with $\mathrm{Re}\,u > x_0$ for some large number $x_0$ to be determined later. Put $v = F^n(u)$ and $x_n = E_\beta^n(x_0)$, where $E_\beta(x) = e^{\beta x}$. Then

$$\mathrm{Re}\,v \geq E_\beta^n(\mathrm{Re}\,u) \geq x_n.$$

A standard argument (cf. Remark 3.3 at the end of this section) using Koebe's distortion theorem shows that for large $u$ and $v$ the branch $\phi_n$ of the inverse function of $F^n$ which satisfies $\phi_n(v) = u$ extends to a univalent map on $B(v, \frac{3}{4}\,\mathrm{Re}\,v)$ and thus has bounded distortion on $Q(v)$. It follows that there exists a constant $K$ such that

$$\mathrm{dens}(\phi_n(Q(v) \cap S), \phi_n(Q(v))) \leq K\,\mathrm{dens}(S, Q(v)) \leq \frac{4K}{N}\delta(x_n). \tag{3.6}$$

Moreover, Koebe's theorem yields that there exist positive constants $\sigma, \tau$ such that if

$$r_n(u) = |\phi_n'(v)| \cdot \operatorname{Re} v = \frac{\operatorname{Re} F^n(u)}{|(F^n)'(u)|},$$

then

$$B(u, \sigma r_n(u)) \subset \phi_n(Q(v)) \subset B(u, \tau r_n(u)). \tag{3.7}$$

It follows from (3.2) that

$$\begin{aligned} \left|(F^n)'(u)\right| &= \left|F'\left(F^{n-1}(u)\right)\right| \cdot \left|\left(F^{n-1}\right)'(u)\right| \\ &\geq \frac{\operatorname{Re} F^n(u) - \log R}{4\pi} \prod_{j=1}^{n-1} \left|F'\left(F^j(u)\right)\right|. \end{aligned}$$

Moreover, (3.2) shows that $\left|F'\left(F^j(u)\right)\right|$ is large for all $j$ if $x_0$ and hence $\operatorname{Re} F^j(u)$ is large. Thus we find that

$$r_n(u) \leq 5\pi \tag{3.8}$$

if $x_0$ is sufficiently large. From (3.6) and (3.7) we can deduce that

$$\operatorname{dens}(F^{-n}(S), B(u, \sigma r_n(u)) \leq \frac{4K}{N} \left(\frac{\tau}{\sigma}\right)^2 \delta(x_n). \tag{3.9}$$

We now fix $w_0$ with $\operatorname{Re} w_0 > 2x_0$ and consider the square $P = Q(w_0)$. Suppose that $n \in \mathbb{N}$ and

$$\operatorname{dens}(T_{n-1}, P) \geq \frac{1}{2}. \tag{3.10}$$

By Lemma 3.1, we can find an at most countable subset $A$ of $T_{n-1} \cap P$ such that the disks $B(u, \sigma r_n(u))$, $u \in A$, cover $T_{n-1} \cap P$, with no point being covered more than $4^4$ times. With

$$\begin{aligned} P' = \Big\{ z \in \mathbb{C} : |\operatorname{Re}(z - w_0)| &< \frac{1}{2} \operatorname{Re} w_0 + 5\pi\sigma, \\ |\operatorname{Im}(z - w_0)| &< \frac{1}{2} \operatorname{Re} w_0 + 5\pi\sigma \Big\} \end{aligned}$$

we have $B(u, \sigma r_n(u)) \subset P'$ for all $u \in A$ by (3.8), and for large $x_0$ we also have $\operatorname{area}(P') \leq 2 \operatorname{area}(P)$.

We now deduce from (3.9) and (3.10) that

$$\begin{aligned}\operatorname{area}(F^{-n}(S)\cap T_{n-1}\cap P) &\leq \operatorname{area}\left(F^{-n}(S)\cap \bigcup_{u\in A} B(u,\sigma r_n(u))\right)\\ &\leq \sum_{u\in A}\operatorname{area}(F^{-n}(S)\cap B(u,\sigma r_n(u)))\\ &\leq \frac{4K}{N}\left(\frac{\tau}{\sigma}\right)^2\delta(x_n)\sum_{u\in A}\operatorname{area}(B(u,\sigma r_n(u)))\\ &\leq \frac{4K}{N}\left(\frac{\tau}{\sigma}\right)^2 4^4\delta(x_n)\operatorname{area}(P')\\ &\leq \frac{8K}{N}\left(\frac{\tau}{\sigma}\right)^2 4^4\delta(x_n)\operatorname{area}(P)\\ &\leq \frac{16K}{N}\left(\frac{\tau}{\sigma}\right)^2 4^4\delta(x_n)\operatorname{area}(T_{n-1}\cap P).\end{aligned}$$

With

$$\eta = \frac{16K}{N}\left(\frac{\tau}{\sigma}\right)^2 4^4$$

we thus have

$$\operatorname{dens}(F^{-n}(S),T_{n-1}\cap P)\leq \eta\delta(x_n).$$

Since $F^{-n}(S)\cap T_{n-1}=T_{n-1}\backslash T_n$ we obtain

$$\operatorname{dens}(T_{n-1}\backslash T_n,T_{n-1}\cap P)\leq \eta\delta(x_n)$$

and thus

$$\operatorname{dens}(T_n,T_{n-1}\cap P)\geq 1-\eta\delta(x_n).$$

Induction shows that

$$\operatorname{dens}(T_n,T_0\cap P)\geq \prod_{k=1}^{n}(1-\eta\delta(x_k)), \tag{3.11}$$

as long as

$$\operatorname{dens}(T_k,P)\geq \frac{1}{2}\quad \text{for } k\leq n-1. \tag{3.12}$$

Now

$$\begin{aligned}\delta(x_n)&=\delta(E_\beta^n(x_0))=\varepsilon(\exp(E_\beta^n(x_0))\\ &\leq \varepsilon(E_\beta^{n+1}(x_0))=\frac{1}{\Phi\left(E_\beta^{n+1}(x_0)\right)}=\frac{1}{\mu^{n+1}\Phi(x_0)}.\end{aligned}$$

We conclude that the infinite product $\prod_{k=1}^{\infty}(1-\eta\delta(x_k))$ converges and by choosing $x_0$ large we may achieve that

$$\prod_{k=1}^{\infty}(1-\eta\delta(x_k))\geq \frac{3}{4}. \tag{3.13}$$

It follows from (3.5) that $\operatorname{dens}(S,P)=\operatorname{dens}(S,Q(w_0))\leq \frac{1}{3}$ if $\operatorname{Re} w_0$ is large enough. Since $T_0=L=\mathbb{C}\setminus S$ we thus find that

$$\operatorname{dens}(T_0,P)=1-\operatorname{dens}(S,P)\geq \frac{2}{3} \tag{3.14}$$

if $\operatorname{Re} w_0$ is large enough.

Suppose now that (3.12) and hence (3.11) holds for some $n \in \mathbb{N}$. Then, since $T_n \subset T_0$,

$$\operatorname{dens}(T_n, P) = \operatorname{dens}(T_n, P \cap T_0) \cdot \operatorname{dens}(T_0, P) \geq \frac{3}{4} \cdot \frac{2}{3} = \frac{1}{2}$$

by (3.13) and (3.14).

Thus (3.12) and hence (3.11) hold with $n-1$ replaced by $n$. Induction thus shows that (3.11) holds for all $n \in \mathbb{N}$. It follows that

$$\operatorname{dens}(T, P \cap T_0) \geq \prod_{k=1}^{\infty} (1 - \eta\delta(x_k)) \geq \frac{3}{4}.$$

In particular, $\operatorname{area}(T) > 0$.

*Remark* 3.1. The essential condition which makes the proof work is (3.5). Such an inequality may hold under hypotheses quite different from ours. Therefore we summarize the hypotheses needed in order to obtain the desired conclusion.

Let $f$ have logarithmic tracts $U_1, \ldots, U_N$ where $1 \leq N \leq \infty$, let $F$ be the function obtained from $f$ by the logarithmic change of variable and put

$$W = \bigcup_{j=1}^{N} \exp^{-1}(U_j), \quad L = \{z \in W : \operatorname{Re} F(z) \geq E_\beta(\operatorname{Re} z)\}$$

and $S = \mathbb{C} \setminus L$, where $\beta > 0$. If there exists a positive, decreasing function $\delta$ satisfying

$$\sum_{n=1}^{\infty} \delta\left(E_\beta^n(x)\right) < \infty \tag{3.15}$$

for some $x > 0$ such that $\operatorname{dens}(S, Q(z)) \leq \delta(\operatorname{Re} z)$ for all $z$, then $I_U(f)$ has positive measure. Here the function $E_\beta(z) = e^{\beta z}$ in (3.15) and in the definition of $L$ can be replaced by another sufficiently fast growing real function.

*Remark* 3.2. For a function $f$ satisfying the hypotheses of Theorem 1.1, the set $\exp(T)$ constructed in the proof is not only contained in the escaping set $I(f)$, but also in the set $A(f)$ of "fast escaping" points introduced in [5]. Thus $A(f)$ also has positive measure if $f$ satisfies the hypotheses of Theorem 1.1. The set $A(f)$ has been studied in a number of papers; see, e. g., [25, 26]

*Remark* 3.3. We used in the proof that the branch $\phi_n$ of the inverse function of $F^n$ which maps $v = F^n(u)$ to $u$ extends to a univalent map on $B(v, \frac{3}{4}\operatorname{Re} v)$. In order to see this we note that if $\phi$ is the branch of $F^{-1}$ which maps $v$ to $F^{n-1}(u)$, then $\phi$ is univalent in $H$ and it follows from (3.1) that

$$\begin{aligned} \operatorname{diam} \phi\left(B\left(v, \frac{3}{4}\operatorname{Re} v\right)\right) &\leq \frac{3}{2}\operatorname{Re} v \max_{w \in B(v, \frac{3}{4}\operatorname{Re} v)} |\phi'(w)| \\ &\leq \frac{3}{2}\operatorname{Re} v \frac{4\pi}{\frac{1}{4}\operatorname{Re} v - \log R} \\ &\leq 48\pi \end{aligned}$$

if $\operatorname{Re} v > 8 \log R$. We conclude that if $u$ and hence $F^{n-1}(u)$ are large enough, then

$$\phi\left(B\left(v, \frac{3}{4}\operatorname{Re} v\right)\right) \subset B\left(F^{n-1}(u), \frac{3}{4}F^{n-1}(u)\right).$$

The above claim now follows by induction.

Essentially the same argument can be found, e.g., in [1, 7]. For entire functions the argument gets much simpler if the postsingular set

$$P(f) = \overline{\bigcup_{n=0}^{\infty} f^n\left(\operatorname{sing}\left(f^{-1}\right)\right)}$$

is bounded. (Here $\operatorname{sing}\left(f^{-1}\right)$ denotes the set of singularities of the inverse function of $f$.) We note that if $f \in B$ and $f_\lambda(z) = \lambda f(z)$, then $P(f_\lambda)$ is bounded for small $\lambda$. A theorem of Rempe [23] implies that there exists $R_\lambda > 0$ such that $f$ and $f_\lambda$ are quasiconformally conjugate on the set $\{z : |f^n(z)| \geq R_\lambda \text{ for all } n \geq 0\}$. Since quasiconformal mappings map sets of positive area to sets of positive area, the conclusion for $f$ follows from that for $f_\lambda$. Thus it actually suffices to consider the special case that $P(f_\lambda)$ is bounded.

*Remark* 3.4. Let $f$ be a function meromorphic in the plane which has $N$ logarithmic tracts $U_1, U_2, \ldots, U_N$. Denote by $T(r,f)$ and $m(r,f)$ the Nevanlinna characteristic and the proximity function of $f$; see [14, 15] for the notation and basic result of Nevanlinna theory. It follows from standard estimates [14, Theorem 7.1] that $\log M_U(r,f) \leq 3m(2r,f)$. Using this it is not difficult to see that the conclusion of Theorem 1.2 holds if

$$\log m(r,f) \leq \left(\frac{N}{2} + \varepsilon(r)\right) \log r$$

and thus, in particular, if

$$\log T(r,f) \leq \left(\frac{N}{2} + \varepsilon(r)\right) \log r.$$

The dynamics of meromorphic functions with logarithmic tracts are studied for example in [2, 8].

## 4. An example

We consider Mittag-Leffler's function

$$E_\alpha(z) = \sum_{n=0}^{\infty} \frac{z^n}{\Gamma(\alpha n + 1)}$$

for a parameter $\alpha \in (0,2)$. It satisfies the following conditions:

(i) $\varrho(E_\alpha) = \frac{1}{\alpha}$
(ii) $E_\alpha$ is bounded in the sector $\left\{re^{it} : r > 0, |t - \pi| \leq \left(1 - \frac{1}{2}\alpha\right)\pi\right\}$
(iii) $E_\alpha \in B$

Properties (i) and (ii) are well-known; see, e.g., [14, pp. 83-86]. Since we could not find a proof of (iii) in the literature, we indicate an argument to prove (iii) below.

It follows from (ii) and (iii) and a theorem of Eremenko and Lyubich [13, Theorem 7] that $\operatorname{area}(I(E_\alpha)) = 0$. Moreover, the arguments yield (cf. [13, Theorem 8]) that if $\lambda > 0$ is chosen so small that the Fatou set of $\lambda E_\alpha$ consists of a single, completely invariant attracting basin, then $\operatorname{area}(J(\lambda E_\alpha)) = 0$.

We see that there exist functions $f \in B$ whose order is arbitrarily close to $\frac{1}{2}$ such that $\operatorname{area}(I(f)) = \operatorname{area}(J(f)) = 0$. Considering $f(z) = E_\alpha(z^N)$ we obtain functions where $A_R$ has $N$ components and where $\varrho(f)$ is close to $\frac{1}{2}N$. Thus the function

$1/\log^m r$ in Theorem 1.1 and the function $\varepsilon(r)$ in Theorem 1.2 cannot be replaced by a positive constant $\varepsilon$.

Let us now prove property (iii). From [14, pp. 84-85] we get the following representation for $E_\alpha$, where $\varrho = 1/\alpha$:

$$E_\alpha(z) = w_1(z) \quad \text{for } \frac{1}{2}\alpha\pi < |\arg(z)| \le \pi, \tag{4.1}$$

$$E_\alpha(z) = w_2(z) + \varrho \exp(z^\varrho) \quad \text{for } |\arg(z)| \le \frac{1}{2}\alpha\pi + \delta, \tag{4.2}$$

where $0 < \delta \le \max\left\{\frac{1}{2}\alpha\pi, \left(1 - \frac{1}{2}\alpha\right)\pi\right\}$ and $w_i(z) = O(1/|z|)$ as $|z| \to \infty$, for $i = 1, 2$. Note that Properties (i) and (ii) follow immediately from (4.1) and (4.2).

To prove Property (iii), put $S_\delta = \{z : |\arg(z)| \le \frac{1}{2}\alpha\pi + \delta\}$. For $z \in S_{\delta/2}$ we have $B(z, |z|\sin(\delta/2)) \subset S_\delta$ and Cauchy's formula yields

$$\begin{aligned} \left|E_\alpha'(z) - \varrho^2 z^{\varrho-1}\exp(z^\varrho)\right| &= |w_2'(z)| \\ &= \left|\frac{1}{2\pi i}\int_{\partial B\left(z,|z|\sin\left(\frac{\delta}{2}\right)\right)} \frac{w_2(\zeta)}{(z-\zeta)^2} d\zeta\right| \\ &= O\left(\frac{1}{|z|^2}\right) \end{aligned} \tag{4.3}$$

as $|z| \to \infty$, uniformly in $z \in S_{\delta/2}$. For $z \in \mathbb{C} \setminus S_{\delta/2}$ we have

$$B(z, |z|\sin(\delta/2)) \subset \mathbb{C} \setminus S_0$$

and in the same way Cauchy's formula yields

$$|E_\alpha'(z)| = |w_1'(z)| = O\left(\frac{1}{|z|^2}\right)$$

as $|z| \to \infty$, uniformly in $\mathbb{C} \setminus S_{\delta/2}$.

We now show that the set of critical values of $E_\alpha$ is bounded. Since $E_\alpha$ is bounded in $\mathbb{C} \setminus S_0$ we have to consider only the critical points in $S_0$. So let $\xi \in S_0$ be a critical point of $E_\alpha$; that is, $E_\alpha'(\xi) = 0$. Then

$$\varrho^2 |\xi|^{\varrho-1} |\exp(\xi^\varrho)| \le \frac{C_1}{|\xi|^2}$$

for some constant $C_1$ by (4.3) and thus

$$|E_\alpha(\xi)| \le \varrho |\exp(\xi^\varrho)| + \frac{C_2}{|\xi|} \le \frac{C_1}{\varrho |\xi|^{\varrho+1}} + \frac{C_2}{|\xi|^2}$$

for some constant $C_2$ by (4.2). It follows that the set of critical values of $E_\alpha$ is bounded. Since $E_\alpha$ has only finitely many asymptotic values by the Denjoy-Carleman-Ahlfors-Theorem, it follows that $f \in B$. (Actually the only asymptotic value of $E_\alpha$ is 0. This can be deduced from (4.1) and (4.2).)

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