KARPIŃSKA'S PARADOX IN DIMENSION THREE

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ABSTRACT. For $0 < \lambda < 1/e$ the Julia set of λe^z is an uncountable union of pairwise disjoint simple curves tending to infinity [Devaney and Krych 1984], the Hausdorff dimension of this set is two [McMullen 1987], but the set of curves without endpoints has Hausdorff dimension one [Karpińska 1999]. We show that these results have three-dimensional analogues when the exponential function is replaced by a quasiregular self-map of $\mathbb{R}^3$ introduced by Zorich.

1. Introduction and main result

Zorich [41] has given an example of a quasiregular map $F : \mathbb{R}^3 \to \mathbb{R}^3\backslash\{0\}$ which in many ways can be considered as a three-dimensional analogue of the exponential map. In fact, the construction is quite flexible and gives a whole class of such maps. It is the purpose of this paper to show that certain results on the dynamics of entire functions of the form $E_\lambda(z) = \lambda e^z$ have counterparts in the context of Zorich maps.

We first describe the results on the dynamics of the functions E_λ that we are concerned with. Then we briefly introduce quasiregular maps and continue with the definition of Zorich maps, before we finally state our results on the dynamics of such maps.

The *Julia set* $J(f)$ of an entire function f is the set of all points in $\mathbb{C}$ where the iterates f^k of f do not form a normal family. For an attracting fixed point ξ of f we call $A(\xi) := \left\{z : \lim_{k\to\infty} f^k(z) = \xi\right\}$ the *attracting basin* of ξ. It is a standard result of complex dynamics that $\partial A(\xi) = J(f)$; see, e.g., [28, Corollary 4.12]. For an introduction to complex dynamics we refer to [4, 28, 38] for rational and to [5, 29] for entire functions.

For $0 < \lambda < 1/e$ the function E_λ has an attracting fixed point $\xi \in \mathbb{R}$. Devaney and Krych [9, p. 50] proved that $J(E_\lambda) = \mathbb{C}\backslash A(\xi)$ and that $J(E_\lambda)$ is a "Cantor set of curves" for such λ. To put this in a precise form we say that a subset H of $\mathbb{C}$ (or $\mathbb{R}^n$) is a *(Devaney) hair* if there exists a homeomorphism $\gamma : [0,\infty) \to H$ such that $\gamma(t) \to \infty$ as $t \to \infty$. We call $\gamma(0)$ the *endpoint* of the hair H. With this terminology we obtain the following result from the work of Devaney and Krych.

Theorem A. *If* $0 < \lambda < 1/e$, *then* $J(E_\lambda)$ *is an uncountable union of pairwise disjoint Devaney hairs.*

Actually the results of Devaney and Krych are much more precise by giving additional information e.g. on the location of the hairs and on the dynamics of E_λ on them, but for simplicity we have restricted ourselves to the above statement.

We denote by $\dim S$ the Hausdorff dimension of a subset S of $\mathbb{C}$ (or $\mathbb{R}^n$). The following result is due to McMullen [27, Theorem 1.2].

Theorem B. *If* $\lambda \in \mathbb{C}\backslash\{0\}$, *then* $\dim J(E_\lambda) = 2$.

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In the situation of Theorem A the union of the hairs thus has Hausdorff dimension 2. Karpińska [21, Theorem 1.1] proved that this changes if one removes the endpoints of the hairs. This surprising result was called Karpińska's paradox by Schleicher and Zimmer [36, p. 380].

Theorem C. *Let* $0 < \lambda < 1/e$ *and let* C_λ *be the set of endpoints of the Devaney hairs that form* $J(E_\lambda)$*. Then* $\dim(J(E_\lambda)\backslash C_\lambda) = 1$.

Of course, it follows from Theorem B and C that $\dim C_\lambda = 2$. This had been proved before also by Karpińska [20, Theorem 1].

The *escaping set* $I(f) := \left\{z : \lim_{k\to\infty} f^k(z) = \infty\right\}$ plays an important role in complex dynamics. It was introduced by Eremenko [11] who showed that $J(f) = \partial I(f)$ for every entire function f. Eremenko and Lyubich [12, Theorem 1] proved that $I(f) \subset J(f) = \overline{I(f)}$ for a large class of functions f, which in particular contains the functions E_λ. McMullen actually proved that $\dim I(E_\lambda) = 2$. We mention that the results of Devaney and Krych [9], together with those of Devaney and Goldberg [8], yield that $J(E_\lambda)\backslash C_\lambda \subset I(E_\lambda)$ while C_λ contains points of both $I(E_\lambda)$ and $\mathbb{C} \setminus I(E_\lambda)$.

There are a large number of papers on dynamics of exponential functions. We refer to a detailed (130 pages) survey by Devaney [7], but note that the papers by Rempe [30] and Schleicher [34] also contain substantial sections devoted to a survey of the area.

We now turn to quasiregular maps. They can be considered as a substitute for holomorphic functions in $\mathbb{R}^n$. Essentially, it is required that infinitesimal balls are mapped to infinitesimal ellipsoids such that the ratio of the largest and the smallest semiaxis is uniformly bounded. We treat this topic rather briefly and refer to Rickman's monograph [31] for a detailed discussion. Let $n \geq 2$ and let $G \subset \mathbb{R}^n$ be a domain. For $1 \leq p < \infty$ the Sobolev space $W^1_{p,loc}(G)$ is defined as the set of functions $f = (f_1, \ldots, f_n) : G \to \mathbb{R}^n$ for which all first order weak partial derivatives $\partial_k f_j$ exist and are locally in L^p. For us only the case $p = n$ will be of interest. We denote the (euclidean) norm of $x \in \mathbb{R}^n$ by $|x|$. A continuous map $f \in W^1_{n,loc}(G)$ is called *quasiregular* if there exists a constant $K_O \geq 1$ such that

$$|Df(x)|^n \leq K_O J_f(x) \quad \text{a.e.,} \tag{1.1}$$

where $Df(x)$ denotes the derivative,

$$|Df(x)| := \sup_{|h|=1} |Df(x)(h)|$$

its norm, and $J_f(x)$ the Jacobian determinant. With

$$\ell(Df(x)) := \inf_{|h|=1} |Df(x)(h)|$$

the condition that (1.1) holds for some $K_O \geq 1$ is equivalent to the condition that

$$J_f(x) \leq K_I \ell(Df(x)) \quad \text{a.e.,} \tag{1.2}$$

for some $K_I \geq 1$. The smallest constants K_O and K_I for which (1.1) and (1.2) hold are called the *outer and inner dilatation* of f and $K := \max\{K_I, K_O\}$ is called the (maximal) *dilatation* of f; see [31] for more details.

An important example of a quasiregular map $F : \mathbb{R}^3 \to \mathbb{R}^3$ was given by Zorich [41, p. 400]; see also [19, §6.5.4] and [31, §I.3.3]. This map can be considered as a three-dimensional analogue of the exponential function. To describe it, we follow [19] and consider the square

$$Q := \left\{(x_1, x_2) \in \mathbb{R}^2 : \ |x_1| \leq 1, |x_2| \leq 1\right\}$$

and the upper hemisphere

$$U := \left\{ (x_1, x_2, x_3) \in \mathbb{R}^3 : \ x_1^2 + x_2^2 + x_3^2 = 1, \ x_3 \geq 0 \right\}.$$

Let $h : Q \to U$ be a bilipschitz map and define

$$F : Q \times \mathbb{R} \to \mathbb{R}^3, \ F(x_1, x_2, x_3) = e^{x_3} h(x_1, x_2).$$

Then F maps the "infinite square beam" $Q \times \mathbb{R}$ to the upper half-space. Repeated reflection along sides of square beams and the (x_1, x_2)-plane yields a map $F : \mathbb{R}^3 \to \mathbb{R}^3$. It turns out that this map F is quasiregular. Its dilatation is bounded in terms of the bilipschitz constant of h. We call a function F defined this way a *Zorich map*.

We note that $F(x_1 + 4, x_2, x_3) = F(x_1, x_2 + 4, x_3) = F(x_1, x_2, x_3)$ for all $x \in \mathbb{R}^3$ so that F is "doubly periodic".

If $DF(x_1, x_2, 0)$ exists, then

$$DF(x_1, x_2, x_3) = e^{x_3} DF(x_1, x_2, 0), \tag{1.3}$$

and this implies that there exist $\alpha, m, M \in \mathbb{R}$ with $0 < \alpha < 1$ and $m < M$ such that

$$|DF(x_1, x_2, x_3)| \leq \alpha \quad \text{a.e. for } x_3 \leq m \tag{1.4}$$

while

$$\ell(DF(x_1, x_2, x_3)) \geq \frac{1}{\alpha} \quad \text{a.e. for } x_3 \geq M. \tag{1.5}$$

We now choose

$$a \geq e^M - m \tag{1.6}$$

and consider the map

$$f_a : \mathbb{R}^3 \to \mathbb{R}^3, \ f_a(x) = F(x) - (0, 0, a).$$

We will use the term Zorich map also for the functions f_a.

We now state the result which can be seen as a three-dimensional analogue of Theorems A, B and C.

Theorem 1. *Let f_a be a Zorich map with parameter a satisfying* (1.6). *Then there exists a unique fixed point $\xi = (\xi_1, \xi_2, \xi_3)$ satisfying $\xi_3 \leq m$, the set*

$$J := \left\{ x \in \mathbb{R}^3 : \ f_a^k(x) \not\to \xi \right\}$$

consists of uncountably many pairwise disjoint hairs, and the set C of endpoints of these hairs has Hausdorff dimension 3 *while $J \backslash C$ has Hausdorff dimension* 1.

For $0 < \lambda < 1/e$ the set C_λ of endpoints of the hairs in $J(E_\lambda)$ can also be characterized as the set of points which are accessible from the attracting basin $A(\xi)$; see [8, Corollary 4.7] and also [21]. It turns out that the situation is different for Zorich maps.

Theorem 2. *Let J be as in Theorem* 1. *Then all points of J are accessible from $\mathbb{R}^3 \backslash J$.*

After some preliminaries in section 2, the parts of Theorem 1 that correspond to Theorems A, B and C will be proved in sections 3, 4 and 5, respectively, while Theorem 2 will be proved in section 6. Some examples of Zorich maps will be discussed in section 7. Besides the techniques introduced by Devaney and Krych [9], Karpińska [21] and McMullen [27] we will also use (in particular in section 3) some methods of Schleicher and Zimmer [36] who obtained analogues of Theorems A and C for general parameter values λ.

We conclude this introduction with a number of remarks.

Remark 1. Instead of the square $Q = \{(x_1, x_2) \in \mathbb{R}^2 : |x_1| \leq 1, |x_2| \leq 1\}$ we could have taken any rectangle $Q = \{(x_1, x_2) \in \mathbb{R}^3 : |x_1| \leq c_1, |x_2| \leq c_2\}$ in the construction of F and f_a. In particular, we may take $c_1 = \frac{1}{2}\pi$ and choose the function $h : Q \to U$ such that $h(x_1, 0) = (\sin x_1, 0, \cos x_1)$. Then $F(x_1, 0, x_3) = (e^{x_3} \sin x_1, 0, e^{x_3} \cos x_1)$. The function F thus leaves the (x_1, x_3)-plane invariant and its restriction to this plane is conjugate to the exponential function in the plane. In fact, with $\phi : \{(x_1, 0, x_3) : x_1, x_3 \in \mathbb{R}\} \to \mathbb{C}$, $\phi(x_1, 0, x_3) = x_3 + ix_1$, we have $\phi \circ F \circ \phi^{-1} = \exp$. This underlines that Zorich maps can be seen as three-dimensional analogues of the exponential function.

Remark 2. Schleicher and Zimmer [36] have shown that $I(E_\lambda)$ consists of pairwise disjoint, injective, unbounded curves for all $\lambda \in \mathbb{C}\backslash\{0\}$. For general Zorich maps $f_a(x) = F(x) - a$ with $a \in \mathbb{R}^3$ the situation is different, since – in contrast to the exponential function – Zorich maps have branch points. In fact, the branch points of F are the edges of the square beam $Q \times \mathbb{R}$ and the lines obtained from these edges by reflection. While it is possible to construct “tails” of hairs by the methods employed in this paper, complications arise when a branch point is mapped onto such a tail by an iterate of F.

We mention that Zorich ([41], see also [31, Corollary III.3.8]) proved a conjecture of Lavrent'ev saying that if $n \geq 3$ and $f : \mathbb{R}^n \to \mathbb{R}^n$ is quasiregular and not bijective, then f has branch points. In fact, the branch set has Hausdorff dimension at least $n - 2$; see [31, p. 12] and [16] for further discussion.

Remark 3. While [36] extends many results from the special case $0 < \lambda < 1/e$ to the general case, some striking differences between these cases were found by Barański, Karpińska and Zdunik [3]: if E_λ has an attracting periodic point of period greater than 1, then the Hausdorff dimension of the boundary of each Fatou component is strictly between 1 and 2.

Remark 4. The exponential maps E_λ have at most one attracting periodic cycle. Zorich maps may have infinitely many attracting fixed points. There may also be saddle points and one-dimensional attractors; see section 7 for more details.

Remark 5. The existence of Devaney hairs is not a special feature of the exponential function. That such hairs exist in large classes of entire functions was already shown by Devaney and Tangerman [10] in 1986, and more recently for considerably wider classes by Barański [1, Theorem C] and by Rottenfußer, Rückert, Rempe and Schleicher [33, Theorem 1.2]. We note that Barański [2] has in fact shown that analogues to Theorems A, B and C hold if the exponential function is replaced by an entire functions of finite order for which the set of singularities of the inverse is bounded.

Remark 6. Karpińska's paradox becomes even more striking in the sine family; that is, for maps f of the form $f(z) = \sin(\alpha z + \beta)$ where $\alpha, \beta \in \mathbb{C}$, $\alpha \neq 0$. Devaney and Tangerman [10, Theorem 4.1] showed that for suitable parameters the Julia set consists of hairs and McMullen [27, Theorem 1.1] proved that the Julia set and the escaping set have positive measure for maps in the sine family. Karpińska [20, Theorem 3] then proved that already the set of endpoints of the hairs has positive measure. A particularly strong form of Karpińska's paradox was obtained by Schleicher [35] for postcritically finite maps in the sine family: the union of the hairs without endpoints has Hausdorff dimension 1, but every point in the plane which is not in this union is the endpoint of at least one hair.

The dynamics of quasiregular analogues of the trigonometric functions, and in particular the analogue of Schleicher's result, will be considered in a forthcoming paper.

Remark 7. Mayer [25] has shown that if $0 < \lambda < 1/e$, then the set C_λ of endpoints of the Devaney hairs is totally disconnected while $C_\lambda \cup \{\infty\}$ is connected. There are a number of other striking topological phenomena connected to the dynamics of exponential functions; see [7] for a survey. It is to be expected that the dynamics of Zorich maps are also interesting from the topological point of view.

Remark 8. A quasiregular map f is called *uniformly quasiregular* if the dilatation of the iterates f^k has an upper bound which does not depend on k. For uniformly quasiregular maps $f : \overline{\mathbb{R}^n} \to \overline{\mathbb{R}^n}$, where $\overline{\mathbb{R}^n} = \mathbb{R}^n \cup \{\infty\}$, an iteration theory in the spirit of Fatou and Julia has been developed by Hinkkanen, Martin, Mayer and others [18, 23, 26]; see [19, Chapter 21] for an introduction. In principle it would also be possible to develop such a theory for uniformly quasiregular maps $f : \mathbb{R}^n \to \mathbb{R}^n$. However, for $n \geq 3$ no examples of such maps which do not extend to quasiregular self-maps of $\overline{\mathbb{R}^n}$ are known.

Uniformly quasiregular self-maps of $\mathbb{R}^2$ (or $\overline{\mathbb{R}^2}$) are quasiconformally conjugate to entire (or rational) maps [15, 17, 22]. The dynamics of quasiregular self-maps of $\overline{\mathbb{R}^2}$ which are not uniformly quasiregular has been studied by Sun and Yang [39, 40]. They show that if the degree of the mapping exceeds its dilatation, then many results of the Fatou-Julia theory still hold. Fletcher and Nicks [14] studied the escaping set of quasiregular self-maps of $\mathbb{R}^n$ which extend to quasiregular self-map of $\overline{\mathbb{R}^n}$ and whose degree is larger than the inner dilatation.

There are only few results on the dynamics of quasiregular self-maps of $\mathbb{R}^n$ with an essential singularity at ∞: in [6] it is proved that $I(f) \neq \emptyset$ for such a map f. In fact, $I(f)$ has an unbounded component. (For entire f this was proved in [32].) We also mention [37] where it is shown that quasiregular self-maps of $\mathbb{R}^n$ with an essential singularity at ∞ have periodic points of all periods greater than 1.

Remark 9. Zorich maps may also be defined in $\mathbb{R}^n$ for $n \geq 4$; see [24]. While it seems that the methods of this paper extend to this more general case, we have restricted to the case $n = 3$ for simplicity. We note that Iwaniec and Martin [19], whose presentation we have followed in the definition of Zorich maps, also confine themselves to the case $n = 3$. Restriction to this case thus allows to use their results directly.

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2. Preliminaries

We suppress the index a and write $f = (f_1, f_2, f_3)$ instead of f_a. For $r = (r_1, r_2) \in \mathbb{Z}^2$ we put

$$P(r) = P(r_1, r_2) := \left\{(x_1, x_2) \in \mathbb{R}^2 : \ |x_1 - 2r_1| < 1, \ |x_2 - 2r_2| < 1\right\}$$

so that $P(0, 0)$ is the interior of Q. For $c \in \mathbb{R}$ we define the half-space

$$H_{>c} := \left\{(x_1, x_2, x_3) \in \mathbb{R}^3 : \ x_3 > c\right\}.$$

The half-spaces $H_{<c}$, $H_{\geq c}$ and $H_{\leq c}$ and the plane $H_{=c}$ are defined analogously. Now F maps $P(r_1, r_2) \times \mathbb{R}$ bijectively onto $H_{>0}$ if $r_1 + r_2$ is even and bijectively onto $H_{<0}$ if $r_1 + r_2$ is odd. Thus f maps $P(r_1, r_2) \times \mathbb{R}$ bijectively onto $H_{>-a}$ or $H_{<-a}$, depending on whether $r_1 + r_2$ is even or odd. For

$$r = (r_1, r_2) \in S := \left\{(s_1, s_2) \in \mathbb{Z}^2 : \ s_1 + s_2 \text{ even}\right\}$$

we define

$$T(r) := P(r) \times (M, \infty).$$

Since $f(P(r) \times \mathbb{R}) = H_{>-a}$ for $r \in S$ and

$$f_3(x_1, x_2, x_3) = e^{x_3} h_3(x_1, x_2) - a \leq e^{x_3} - e^M + m \leq m < M \quad \text{for } x_3 \leq M \tag{2.1}$$

and hence $f(P(r)\times(-\infty,M])\subset H_{<M}$ we see that $f(T(r))\supset H_{\geq M}$. Thus there exists a branch $\Lambda^r : H_{\geq M}\to T(r)$ of the inverse function of f. With $\Lambda := \Lambda^{(0,0)}$ we have

$$\Lambda^{(r_1,r_2)}(x) = \Lambda(x) + (2r_1, 2r_2, 0)$$

for all $x\in H_{\geq M}$ and all $r\in S$.

We shall need some estimates for the derivative $D\Lambda^r$. Since $D\Lambda^r(x) = D\Lambda(x)$ whenever these derivatives exist it suffices to obtain these estimates for $D\Lambda$. First we note that

$$D\Lambda(x) = Df(\Lambda(x))^{-1} \tag{2.2}$$

for $x\in H_{\geq M}$. Since $\Lambda(x)\in T(0,0)\subset H_{\geq M}$ for $x\in H_{\geq M}$ and since $DF = Df$ we deduce from (1.5) that $|D\Lambda(x)|\leq\alpha$ a.e. for $x\in H_{\geq M}$. This implies that

$$|\Lambda(x)-\Lambda(y)|\leq|x-y|\operatorname*{ess\,sup}_{z\in[x,y]}|D\Lambda(z)|\leq\alpha|x-y|\quad\text{for } x,y\in H_{\geq M}. \tag{2.3}$$

Next we note that (1.3) implies that there exist positive constants c_1 and c_2 such that

$$c_1 e^{x_3}\leq\ell\left(Df(x_1,x_2,x_3)\right)\leq|Df(x_1,x_2,x_3)|\leq c_2 e^{x_3}\quad\text{a.e.} \tag{2.4}$$

Noting that

$$|f(y_1,y_2,y_3)|-a\leq|F(y_1,y_2,y_3)| = e^{y_3}\leq|f(y_1,y_2,y_3)|+a \tag{2.5}$$

for all $(y_1,y_2,y_3)\in\mathbb{R}^3$ we deduce from (2.2) and (2.4) that

$$\ell\left(D\Lambda(x)\right)\geq\frac{1}{|DF(\Lambda(x))|}\geq\frac{1}{c_2\exp\left(\Lambda_3(x)\right)}\geq\frac{1}{c_2\left(|f\left(\Lambda(x)\right)|+a\right)} = \frac{1}{c_2(|x|+a)}\quad\text{a.e.}$$

Thus there exists $c_3>0$ such that

$$\ell\left(D\Lambda(x)\right)\geq\frac{c_3}{|x|}\quad\text{a.e.} \tag{2.6}$$

for $x\in H_{\geq M}$. Similarly we have

$$|D\Lambda(x)|\leq\frac{c_4}{|x|}\quad\text{a.e.} \tag{2.7}$$

for some constant $c_4>0$, as well as

$$\frac{c_5}{|x|^3}\leq J_\Lambda(x)\leq\frac{c_6}{|x|^3}\quad\text{a.e.} \tag{2.8}$$

where $c_5, c_6>0$.

Let now $x,y\in H_{\geq M}$. Then x and y can be connected by a path γ in

$$H_{\geq M}\cap\left\{z\in\mathbb{R}^3:\ |z|\geq\min\{|x|,|y|\}\right\}$$

whose length is not greater than $\pi|x-y|$. Together with (2.7) this yields

$$|\Lambda(x)-\Lambda(y)|\leq\pi|x-y|\operatorname*{ess\,sup}_{z\in\gamma}|D\Lambda(z)|\leq c_4\pi\frac{|x-y|}{\min\{|x|,|y|\}} \tag{2.9}$$

Next we note that there exists a unique point $(v_1,v_2)\in Q$ such that $h(v_1,v_2) = (0,0,1)$. Then $F(v_1,v_2,x_3) = (0,0,e^{x_3})$ and hence $f(v_1,v_2,x_3) = (0,0,e^{x_3}-a)$. It follows that if $r = (r_1,r_2)\in S$, then

$$\Lambda^r(0,0,e^{x_3}-a) = (v_1+2r_1, v_2+2r_2, x_3)$$

for $x_3\in\mathbb{R}$ and thus

$$\Lambda^r(0,0,t) = (v_1+2r_1, v_2+2r_2, \log(t+a)) \tag{2.10}$$

for $t\geq M$.

As in [36] we will consider the function

$$E : [0, \infty) \to [0, \infty), \ E(t) = e^t - 1.$$

We have $E(0) = 0$ while $\lim_{k\to\infty} E^k(t) = \infty$ if $t > 0$. Later we will use that if $b > 1$, then

$$\log\left(E^{k+1}(t) + b\right) = \log\left(\exp(E^k(t)) - 1 + b\right) = E^k(t) + \log\left(1 + \frac{b-1}{\exp(E^k(t))}\right)$$

so that

$$\log\left(E^{k+1}(t) + b\right) = E^k(t) + R_k(t) \quad \text{with } 0 \le R_k(t) \le \log b. \tag{2.11}$$

We also note that if $0 < t' < t'' < \infty$, then

$$\lim_{k\to\infty}\left(E^k(t'') - E^k(t')\right) = \infty \quad \text{and} \quad \lim_{k\to\infty}\frac{E^k(t'')}{E^k(t')} = \infty. \tag{2.12}$$

3. Construction of the hairs

In this section we prove that the fixed point ξ as given in the theorem exists and that J is a union of uncountably many pairwise disjoint hairs. We thus prove the part of the conclusion of Theorem 1 that is the analogue of Theorem A.

We first note that $f(H_{\le M}) \subset H_{\le m}$ by (2.1). Moreover,

$$|f(x) - f(y)| \le |x - y| \operatorname*{ess\,sup}_{z\in[x,y]} |Df(z)| \le \alpha|x - y|$$

for $x, y \in H_{\le m}$ by (1.4). Banach's fixed point theorem now implies that there exists a unique fixed point $\xi \in H_{\le m}$ and that $f^n(x) \to \xi$ as $n \to \infty$ for all $x \in H_{\le M}$.

If $r_1 + r_2$ is odd, then $f(P(r_1, r_2) \times \mathbb{R}) = H_{<-a}$. Since $-a < m < M$ we deduce that f maps the closure of $P(r_1, r_2) \times \mathbb{R}$ into $H_{\le M}$ if $r_1 + r_2$ is odd. Thus

$$J \subset \bigcup_{r\in S} T(r).$$

As in the case of exponential maps we associate to each $x \in J$ a sequence

$$\underline{s}(x) = s_0 s_1 s_2 \ldots = (s_k)_{k\ge 0}$$

in S, where $s_k = (s_{k,1}, s_{k,2}) \in S$ is chosen such that $f^k(x) \in T(s_k)$ for all $k \ge 0$. The sequence $\underline{s}(x)$ is called the *itinerary* (or *external address*) of x. We denote the set of all sequences $\underline{s} : \mathbb{N} \cup \{0\} \to S$ by Σ. We say that $\underline{s} = (s_k)_{k\ge 0} \in \Sigma$ is *admissible* (or *exponentially bounded*) if there exists $t > 0$ such that

$$\limsup_{k\to\infty} \frac{|s_k|}{E^k(t)} < \infty. \tag{3.1}$$

Here $|s_k| = |(s_{k,1}, s_{k,2})| = \sqrt{s_{k,1}^2 + s_{k,2}^2}$ is the euclidean norm of s_k.

With these notations we have the following two propositions.

Proposition 3.1. *Let $x \in J$. Then $\underline{s}(x)$ is admissible.*

Proposition 3.2. *Let $\underline{s} \in \Sigma$ be admissible. Then $\{x \in J : \underline{s}(x) = \underline{s}\}$ is a hair.*

Since $\{x \in J : \underline{s}(x) = \underline{s}\} \cap \{x \in J : \underline{s}(x) = \underline{s}'\} = \emptyset$ for $\underline{s}, \underline{s}' \in \Sigma$, $\underline{s} \ne \underline{s}'$, it follows from these two propositions that J is a union of pairwise disjoint hairs. Moreover, the set of admissible sequences – and thus the set of hairs – is easily seen to be uncountable.

The proof of Proposition 3.1 is straightforward, using exactly the same reasoning as in the case of exponential maps. Therefore we omit it here.

For the proof of Proposition 3.2 we suitably modify the arguments of Schleicher and Zimmer [36]. We fix an admissible sequence $\underline{s}$ and denote by $t_{\underline{s}}$ the infimum of all $t > 0$ for which (3.1) holds. It follows from (2.12) that

$$\limsup_{k\to\infty} \frac{|s_k|}{E^k(t)} = \infty \quad \text{for } 0 < t < t_{\underline{s}} \tag{3.2}$$

and

$$\lim_{k\to\infty} \frac{|s_k|}{E^k(t)} = 0 \quad \text{for } t > t_{\underline{s}}. \tag{3.3}$$

Choosing $t_k \in [0,\infty)$ such that $2|s_k| = E^k(t_k)$ and putting $\tau_k := \sup_{j\geq k} t_j$ we have

$$t_{\underline{s}} = \limsup_{k\to\infty} t_k = \lim_{k\to\infty} \tau_k.$$

We use the abbreviation

$$L_k := \Lambda^{s_k} = \Lambda^{(s_{k,1},s_{k,2})}.$$

For $k \geq 0$ we define

$$g_k : [0,\infty) \to H_{\geq M},\ g_k(t) = (L_0 \circ L_1 \circ \ldots \circ L_k)(0,0,E^{k+1}(t)+M).$$

Lemma 3.1. *The sequence (g_k) converges locally uniformly on $(t_{\underline{s}},\infty)$.*

Proof. It follows from (2.10) and (2.11) that

$$\begin{aligned} L_k(0,0,E^{k+1}(t)+M) &= (v_1 + 2s_{k,1}, v_2 + 2s_{k,2}, \log(E^{k+1}(t)+M+a)) \\ &= (v_1 + 2s_{k,1}, v_2 + 2s_{k,2}, E^k(t) + R_k(t)) \end{aligned} \tag{3.4}$$

where $0 \leq R_k(t) \leq \log(M+a)$. Since $|v_1| \leq 1$ and $|v_2| \leq 1$ this yields

$$\left|L_k(0,0,E^{k+1}(t)+M)\right| \geq \left|(2s_{k,1}, 2s_{k,2}, E^k(t))\right| - c_7$$

with $c_7 := 2 + \log(M+a)$. Since $L_k(H_{\geq M}) \subset H_{\geq M}$ this implies that

$$\left|L_k(0,0,E^{k+1}(t)+M)\right| \geq \max\{2|s_k| - c_7, E^k(t) - c_7, M\} \tag{3.5}$$

Also, it follows from (3.4) that

$$\left|L_k(0,0,E^{k+1}(t)+M) - (0,0,E^k(t)+M)\right| \leq 2|s_k| + c_7 + M. \tag{3.6}$$

Applying (2.9) with $x = L_k(0,0,E^{k+1}(t)+M)$, $y = (0,0,E^k(t)+M)$ and $\Lambda = L_{k-1}$ and noting that $E^k(t) \geq 2|s_k|$ for $t \geq t_k$ we deduce from (3.5) and (3.6) that if $t \geq t_k$, then

$$\left|L_{k-1}(L_k(0,0,E^{k+1}(t)+M)) - L_{k-1}(0,0,E^k(t)+M)\right| \leq c_4\pi \frac{E^k(t)+c_7+M}{\max\{E^k(t)-c_7,M\}} \leq c_8$$

for some constant c_8. Now (2.3) gives

$$\left|L_{k-2}(L_{k-1}(L_k(0,0,E^{k+1}(t)+M))) - L_{k-2}(L_{k-1}(0,0,E^k(t)))\right| \leq \alpha c_8$$

and induction yields

$$|g_k(t) - g_{k-1}(t)| \leq \alpha^{k-1} c_8 \quad \text{for } t \geq t_k. \tag{3.7}$$

In particular, $|g_k(t) - g_{k-1}(t)| \leq \alpha^{k-1} c_8$ if $k \geq j$ and $t \geq \tau_j$. As $\lim_{j\to\infty} \tau_j = t_{\underline{s}}$ this implies that (g_k) converges locally uniformly on $(t_{\underline{s}},\infty)$. ☐

Define $g : (t_{\underline{s}}, \infty) \to H_{\geq M}$ by $g(t) = \lim_{k\to\infty} g_k(t)$. Then g is continuous and (3.7) yields

$$|g(t) - g_{k-1}(t)| \leq \frac{\alpha^{k-1}c_8}{1-\alpha} \quad \text{for } t \geq \tau_k. \tag{3.8}$$

We also have

$$|g_l(t) - g_{k-1}(t)| \leq \frac{\alpha^{k-1}c_8}{1-\alpha} \quad \text{for } t \geq \tau_k \text{ and } l \geq k. \tag{3.9}$$

In particular, (3.8) yields

$$|g(t) - g_0(t)| \leq \frac{c_8}{1-\alpha} \quad \text{for } t \geq \tau_1. \tag{3.10}$$

Since

$$g_0(t) = L_0(0,0,E(t)+M) = (v_1 + 2s_{0,1}, v_2 + 2s_{0,2}, \log(E(t)+M+a))$$

by (2.10) we deduce from (2.11) that

$$|g_0(t) - (2s_{0,1}, 2s_{0,2}, t)| \leq 2 + \log(M+a) = c_7.$$

Combining this with (3.10) we obtain

$$|g(t) - (2s_{0,1}, 2s_{0,2}, t)| \leq c_9 \quad \text{for } t \geq \tau_1, \tag{3.11}$$

where $c_9 := c_7 + c_8/(1-\alpha)$.

Lemma 3.2. *The sequence (g_k) has a subsequence which converges uniformly on $[t_{\underline{s}}, \infty)$ and thus g extends to a continuous map $g : [t_{\underline{s}}, \infty) \to H_{\geq M}$.*

Proof. The conclusion follows immediately from (3.7) if $t_k \leq t_{\underline{s}}$ for all large k and thus $\tau_k = t_{\underline{s}}$ for large k, since then (g_k) converges uniformly on $[t_{\underline{s}}, \infty)$. We may thus assume that there exist arbitrarily large k for which $t_k > t_{\underline{s}}$. Then $\tau_k > t_{\underline{s}}$ for all k and there exists an increasing sequence (k_j) such that $\tau_k = t_{k_j}$ for $k_{j-1} < k \leq k_j$. Thus $\tau_{k_j+1} = \tau_{k_{j+1}}$ and (3.9) implies that

$$\left|g_{k_{j+1}}\left(\tau_{k_{j+1}}\right) - g_{k_j}\left(\tau_{k_{j+1}}\right)\right| \leq \frac{\alpha^{k_j}c_8}{1-\alpha} \leq \frac{\alpha^j c_8}{1-\alpha}. \tag{3.12}$$

It follows from (3.4) that if $0 \leq t \leq t_k$, then

$$\begin{aligned} & \left|L_k(0,0,E^{k+1}(t_k)+M) - L_k(0,0,E^{k+1}(t)+M)\right| \\ = & \left|\left(0,0,E^k(t_k) - E^k(t) + R_k(t_k) - R_k(t)\right)\right| \\ \leq & E_k(t_k) + 2\log(M+a) \\ = & 2|s_k| + 2\log(M+a). \end{aligned}$$

Combining this with (2.9) and (3.5) we find that

$$\begin{aligned} & \left|L_{k-1}(L_k(0,0,E^{k+1}(t_k)+M)) - L_{k-1}(L_k(0,0,E^{k+1}(t)+M))\right| \\ \leq & c_4\pi \frac{2|s_k| + 2\log(M+a)}{\max\{2|s_k| - c_7, M\}} \\ \leq & c_{10} \quad \text{for } 0 \leq t \leq t_k \end{aligned}$$

with a constant c_{10} and thus (2.3) yields

$$|g_k(t_k) - g_k(t)| \leq \alpha^{k-1}c_{10} \quad \text{for } 0 \leq t \leq t_k.$$

Hence

$$\left|g_{k_{j+1}}(\tau_{k_{j+1}}) - g_{k_{j+1}}(t)\right| \leq \alpha^{k_{j+1}-1}c_{10} \leq \alpha^j c_{10} \quad \text{for } 0 \leq t \leq \tau_{k_{j+1}}$$

and, since $\tau_{k_{j+1}} \le \tau_{k_j}$, also

$$\left|g_{k_j}(\tau_{k_j}) - g_{k_j}(t)\right| \le \alpha^{j-1} c_{10} \quad \text{for } 0 \le t \le \tau_{k_{j+1}}.$$

In particular,

$$\left|g_{k_j}(\tau_{k_j}) - g_{k_j}(\tau_{k_{j+1}})\right| \le \alpha^{j-1} c_{10}.$$

Combination of the last three inequalities with (3.12) yields

$$\begin{aligned}\left|g_{k_{j+1}}(t) - g_{k_j}(t)\right| &\le \left|g_{k_{j+1}}(t) - g_{k_{j+1}}(\tau_{k_{j+1}})\right| + \left|g_{k_{j+1}}(\tau_{k_{j+1}}) - g_{k_j}(\tau_{k_{j+1}})\right| \\ &\quad + \left|g_{k_j}(\tau_{k_{j+1}}) - g_{k_j}(\tau_{k_j})\right| + \left|g_{k_j}(\tau_{k_j}) - g_{k_j}(t)\right| \\ &\le \left(3c_{10} + \frac{\alpha}{1-\alpha} c_8\right) \alpha^{j-1} \quad \text{for } 0 \le t \le \tau_{k_{j+1}}.\end{aligned}$$

On the other hand, (3.9) implies that

$$\left|g_{k_{j+1}}(t) - g_{k_j}(t)\right| \le \alpha^{k_j - 1} \frac{c_8}{1-\alpha} \le \alpha^{j-1} \frac{c_8}{1-\alpha} \quad \text{for } t \ge \tau_{k_j+1} = \tau_{k_{j+1}}.$$

The last two inequalities show that $\left(g_{k_j}\right)$ converges uniformly on $[0, \infty)$ and thus in particular on $[t_{\underline{s}}, \infty)$. ☐

We note that g_k and g depend on $\underline{s}$. Using the self-explanatory notation $g_{\underline{s},k}$ and $g_{\underline{s}}$ we find that

$$f(g_{\underline{s},k}(t)) = g_{\sigma(\underline{s}),k-1}(E(t))$$

where $\sigma : \Sigma \to \Sigma$, $\sigma(s_0 s_1 s_2 \dots) = s_1 s_2 s_3 \dots$, is the shift map. Thus $f(g_{\underline{s}}(t)) = g_{\sigma(\underline{s})}(E(t))$ and

$$f^k(g_{\underline{s}}(t)) = g_{\sigma^k(\underline{s})}(E^k(t)) \tag{3.13}$$

for $t \ge 0$ and $k \in \mathbb{N}$. Similarly, the sequences (t_k) and (τ_k) depend on $\underline{s}$. Using the notation $t_{\underline{s},k}$ and $\tau_{\underline{s},k}$ a more precise formulation of (3.11) would thus be

$$|g_{\underline{s}}(t) - (2s_{0,1}, 2s_{0,2}, t)| \le c_9 \quad \text{for } t \ge \tau_{\underline{s},1}. \tag{3.14}$$

We also note that

$$E^{j+k}(t_{\underline{s},j+k}) = 2|s_{j+k}| = E^j(t_{\sigma^k(\underline{s}),j})$$

and hence

$$E^k(t_{\underline{s},j+k}) = t_{\sigma^k(\underline{s}),j} \quad \text{and} \quad E^k(\tau_{\underline{s},j+k}) = \tau_{\sigma^k(\underline{s}),j}$$

for $j, k \ge 0$.

For $x = x_0 \in J$ and $k \ge 0$ we put $x_k = (x_{k,1}, x_{k,2}, x_{k,3}) := f^k(x)$. Similarly we write $y_k = (y_{k,1}, y_{k,2}, y_{k,3}) := f^k(y)$.

Lemma 3.3. *There exists a positive constant H such that if $x, y \in J$ with $\underline{s}(x) = \underline{s}(y)$ and if $y_{k,3} \ge x_{k,3} + H$ for some $k \in \mathbb{N}$, then $y_{j,3} \ge x_{j,3} + H$ for all $j > k$.*

Proof. Since $\underline{s}(x) = \underline{s}(y)$ we have $|(y_{k,1}, y_{k,2}) - (x_{k,1}, x_{k,2})| \le 4$. Thus

$$y_{k+1,3} \ge |y_{k+1}| - |(y_{k,1}, y_{k,2})| \ge e^{y_{k,3}} - a - |(x_{k,1}, x_{k,2})| - 4 \ge e^H e^{x_{k,3}} - a - |x_{k+1}| - 4$$

by (2.5). If $|x_{k+1}| \ge 2a$, then $\exp(x_{k,3}) \ge |x_{k+1}| - a \ge \frac{1}{2}|x_{k+1}|$ by (2.5) and thus

$$y_{k+1,3} \ge \left(\tfrac{1}{2} e^H - 1\right) |x_{k+1}| - a - 4.$$

If $|x_{k+1}| < 2a$, then

$$y_{k+1,3} \ge e^H e^M - 3a - 4 \ge e^H e^M \frac{|x_{k+1}|}{2a} - 3a - 4.$$

In both cases we we obtain $y_{k+1,3} > |x_{k+1}| + H \geq x_{k+1,3} + H$ if H is chosen sufficiently large. The conclusion follows by induction. □

Lemma 3.4. *$g_{\underline{s}}$ is injective on $[t_{\underline{s}}, \infty)$.*

Proof. Let $t_{\underline{s}} < v < w$ and and $k \in \mathbb{N}$. Put

$$y_k := f^k\left(g_{\underline{s}}(v)\right) = g_{\sigma^k(\underline{s})}\left(E^k(v)\right) \quad \text{and} \quad z_k := f^k\left(g_{\underline{s}}(w)\right) = g_{\sigma^k(\underline{s})}\left(E^k(w)\right).$$

For large k we have $v \geq \tau_{\underline{s},k+1}$ and thus $E^k(w) \geq E^k(v) \geq E^k(\tau_{\underline{s},k+1}) = \tau_{\sigma^k(\underline{s}),1}$. It follows from (3.14) that

$$\left|y_k - (2s_{k,1}, 2s_{k,2}, E^k(v))\right| \leq c_9 \quad \text{and} \quad \left|z_k - (2s_{k,1}, 2s_{k,2}, E^k(w))\right| \leq c_9$$

for such k. In particular, $\left|y_{k,3} - E^k(v)\right| \leq c_9$ and $\left|z_{k,3} - E^k(w)\right| \leq c_9$ for large k. From (2.12) we deduce that $E^k(w) > E^k(v) + H + 2c_9$ and thus

$$z_{k,3} > y_{k,3} + H \tag{3.15}$$

for large k. In particular, $z_k \neq y_k$ and thus $g_{\underline{s}}(w) \neq g_{\underline{s}}(v)$. Hence $g_{\underline{s}}$ is injective on $(t_{\underline{s}}, \infty)$.

Suppose now that there exists $k \in \mathbb{N}$ such that $y_{k,3} \geq z_{k,3} + H$. Lemma 3.3 implies that $y_{j,3} \geq z_{j,3} + H$ for all $j > k$. This contradicts (3.15) and thus we have $y_{k,3} < z_{k,3} + H$ for all $k \in \mathbb{N}$. The same argument shows that if $t_{\underline{s}} < u < v$ and $x_k := f^k(g_{\underline{s}}(u))$, then $x_{k,3} < y_{k,3} + H$ for all $k \in \mathbb{N}$. Let $x'_k := f^k(g_{\underline{s}}(t_{\underline{s}}))$. With $u \to t_{\underline{s}}$ we obtain $x'_{k,3} \leq y_{k,3} + H$ for all $k \in \mathbb{N}$. Using (3.15) we obtain $x'_{k,3} < z_{k,3}$ and thus $g_{\underline{s}}(t_{\underline{s}}) \neq g_{\underline{s}}(w)$. Hence $g_{\underline{s}}$ is in fact injective on $[t_{\underline{s}}, \infty)$. □

Lemma 3.5. *Let $x \in J$. Then $x_{k,3} \geq E^k(t_{\underline{s}(x)})$ for $k \geq 0$.*

Proof. We put $\underline{s} = (s_k) := \underline{s}(x)$. As the conclusion is trivial for $t_{\underline{s}} = 0$ we may assume that $t_{\underline{s}} > 0$. Since $a \geq e^M - M \geq 1$ we have $x_{k+1,3} \leq \exp(x_{k,3}) - a \leq \exp(x_{k,3}) - 1 = E(x_{k,3})$ and thus

$$x_{n,3} \leq E^{n-k}(x_{k,3}) \tag{3.16}$$

for $n \geq k$. Because $|x_{k,1} - 2s_{k,1}| \leq 1$ and $|x_{k,2} - 2s_{k,2}| \leq 1$ we deduce from (2.5) that

$$2|s_k| \leq |x_k| + 2 \leq \exp(x_{k-1,3}) + a + 2 = E(x_{k-1,3}) + a + 3 \tag{3.17}$$

for all $k \geq 0$. Let now $\delta \in (0, t_{\underline{s}})$. Then there exists arbitrarily large l with

$$2|s_l| - a - 3 \geq E^l(t_{\underline{s}} - \delta). \tag{3.18}$$

In particular, given $k \geq 0$ there exists $l > k$ with this property. Combining (3.17) and (3.18) we see that $E^l(t_{\underline{s}} - \delta) \leq E(x_{l-1,3})$. Hence

$$E^{l-k}\left(E^k(t_{\underline{s}} - \delta)\right) = E^l(t_{\underline{s}} - \delta) \leq E(x_{l-1,3}) \leq E^{l-k}(x_{k,3})$$

by (3.16) and thus

$$E^k(t_{\underline{s}} - \delta) \leq x_{k,3}.$$

As δ can be chosen arbitrarily small, the conclusion follows. □

Lemma 3.6. *Let $x \in J$. Then there exist $t \geq t_{\underline{s}(x)}$ with $x = g_{\underline{s}(x)}(t)$.*

Proof. We again put $\underline{s} = (s_k) := \underline{s}(x)$ and note that

$$L_k(f^{k+1}(x)) = f^k(x) = (x_{k,1}, x_{k,2}, x_{k,3})$$

where $|x_{k,1} - 2s_{k,1}| \leq 1$, $|x_{k,2} - 2s_{k,2}| \leq 1$ and $x_{k,3} \geq M$. Define u_k by $x_{k,3} = E^k(u_k)$. Lemma 3.5 implies that $u_k \geq t_{\underline{s}}$. Moreover, it follows from (3.16) that (u_k) is decreasing and hence convergent.

Using (3.4) we find that

$$\begin{aligned} & \left|L_k(f^{k+1}(x)) - L_k((0,0,M+E^{k+1}(u_k)))\right| \\ = & \ |(x_{k,1}-2s_{k,1}-v_1, x_{k,2}-2s_{k,2}-v_2, -R_k(u_k))| \\ \leq & \ 4+\log(M+a). \end{aligned} \tag{3.19}$$

Since $x = (L_0 \circ L_1 \circ \ldots \circ L_k)(f^{k+1}(x))$ we deduce similarly as in the proofs of Lemma 3.1 and Lemma 3.4 from (2.3) and (3.19) that

$$|x - g_k(u_k)| \leq \alpha^k (4 + \log(M+a))$$

so that $x = \lim_{k\to\infty} g_k(u_k)$. With $t := \lim_{k\to\infty} u_k$ we obtain $x = g_{\underline{s}}(t)$. □

Proposition 3.2 follows immediately from Lemmas 3.1–3.6. As already mentioned it yields together with Proposition 3.1 that J is a union of pairwise disjoint Devaney hairs. In fact, if Σ' denotes the set of admissible sequences we have

$$J = \bigcup_{\underline{s}\in\Sigma'} g_{\underline{s}}\left([t_{\underline{s}}, \infty)\right) \quad \text{and} \quad C = \left\{g_{\underline{s}}\left(t_{\underline{s}}\right) : \ \underline{s} \in \Sigma'\right\}.$$

4. The Hausdorff dimension of the hairs

In this section we prove that $\dim J = 3$, thereby establishing the analogue of Theorem B. Following McMullen [27] we consider for $k \in \mathbb{N}$ a finite collection A_k of disjoint compact subsets of $\mathbb{R}^n$ such that the following two conditions are satisfied:

(a) every element of A_{k+1} is contained in a unique element of A_k;
(b) every element of A_k contains at least one element of A_{k+1}.

Denote by $\overline{A}_k$ the union of all elements of A_k and put

$$A := \bigcap_{k=1}^{\infty} \overline{A}_k.$$

Suppose that (Δ_k) and (d_k) are sequences of positive real numbers such that if $B \in A_k$, then

$$\operatorname{dens}\left(\overline{A}_{k+1}, B\right) := \frac{\operatorname{area}\left(\overline{A}_{k+1} \cap B\right)}{\operatorname{area}(B)} \geq \Delta_k$$

and

$$\operatorname{diam} B \leq d_k.$$

Then we have the following result [27, Proposition 2.2].

Lemma 4.1. *Let A, A_k, Δ_k and d_k be as above. Then*

$$\limsup_{k\to\infty} \frac{\sum_{j=1}^{k} |\log \Delta_j|}{|\log d_k|} \geq n - \dim A.$$

The construction of the sets A_k will be very similar to that in [27], but in contrast to [27] we will not have $\Delta_k \geq \Delta$ for some $\Delta > 0$ and all k, but $\lim_{k\to\infty} \Delta_k = 0$. However, the sequence (Δ_k) will tend to 0 much more slowly than (d_k) so that Lemma 4.1 can still be applied.

To begin the construction we note that there exists $q \in (0,1)$ such that

$$\left\{(x_1,x_2,x_3) \in \mathbb{R}^3 : \ x_1^2+x_2^2+x_3^2 = 1, x_3 \geq \tfrac{1}{2}\right\} \subset h\left(\left\{(x_1,x_2) \in \mathbb{R}^2 : \ |x_1| \leq q, |x_2| \leq q\right\}\right).$$

For $r = (r_1, r_2) \in S$ and $\ell \in \mathbb{N}$ we consider the box

$$R(r,\ell) = R(r_1,r_2,\ell) = \left\{x \in \mathbb{R}^3 : \ |x_1 - 2r_1| \leq q, \ |x_2 - 2r_2| \leq q, \ \ell \leq x_3 \leq \ell + \tfrac{3}{4}\right\}$$

and we denote by U the collection of all $R(r,\ell)$; that is,

$$U=\{R(r,\ell):\ r\in S,\ \ell\in\mathbb{N}\}.$$

We note that $F(R(r,\ell))$ does not depend on $r\in S$ and that

$$\left\{x\in\mathbb{R}^3:\ e^\ell\leq|x|\leq e^{3/4}e^\ell,\ x_3\geq\tfrac{1}{2}|x|\right\}\subset F(R(r,\ell))\subset\left\{x\in\mathbb{R}^3:\ e^\ell\leq|x|\leq e^{3/4}e^\ell\right\}.$$

Since $2<\exp\left(\frac{3}{4}\right)<3$ this yields

$$\left\{x\in\mathbb{R}^3:\ \tfrac{3}{2}e^\ell\leq|x|\leq 2e^\ell,\ x_3\geq\tfrac{1}{2}|x|\right\}\subset f(R(r,\ell))\subset\left\{x\in\mathbb{R}^3:\ \tfrac{1}{2}e^\ell\leq|x|\leq 3e^\ell\right\}\tag{4.1}$$

if ℓ is large. We put

$$U(\ell):=\{R\in U:\ R\subset f(R(r,\ell))\}\quad\text{and}\quad\overline{U}(\ell):=\bigcup_{R\in U(\ell)}R,$$

noting that this definition does not depend on $r\in S$. We deduce from (4.1) that there exists a positive constant δ such that

$$\operatorname{dens}\left(\overline{U}(\ell),f(R(r,\ell))\right)\geq\delta\tag{4.2}$$

for all sufficiently large ℓ.

Let now ℓ_0 be so large that (4.1) and (4.2) hold for $\ell\geq\ell_0$. We put $A_0:=\{R(0,0,\ell_0)\}$. We will construct the sets A_k inductively such that if $B\in A_k$, then there exist $r_j=(r_{j,1},r_{j,2})\in S$ and $\ell_j\in\mathbb{N}$, for $j=1,2,\dots,k$, such that

$$f^k(B)=R(r_k,\ell_k),$$

$$f^j(B)\subset R(r_j,\ell_j)\quad\text{for } j=1,2,\dots,k-1$$

and

$$\ell_j\geq\tfrac{1}{2}\exp(\ell_{j-1})\geq\ell_{j-1}\geq\ell_0\quad\text{for } j=1,2,\dots,k.$$

Assuming that A_k has been constructed and B is as above, we put

$$A_{k+1}(B):=\{(L_0\circ L_1\circ\dots\circ L_k)(R):\ R\in U(\ell_k)\}$$

and

$$A_{k+1}:=\bigcup_{B\in A_k}A_{k+1}(B),$$

with $L_j=\Lambda^{r_j}$ as in section 3. Then A_{k+1} has the required properties.

It follows from (2.8), (4.1) and (4.2) that if $B\in A_k$ with r_j and ℓ_j as above, then

$$\operatorname{dens}\left(L_k(\overline{U}(\ell_k)),R(r_k,\ell_k)\right)=\operatorname{dens}\left(L_k(\overline{U}(\ell_k)),L_k(f(R(r_k,\ell_k)))\right)\geq\eta\delta$$

where $\eta:=c_5/(216c_6)$. Since

$$(L_j\circ\dots\circ L_k)(f(R(r_k,\ell_k)))\subset R(r_j,\ell_j)\subset\left\{x\in\mathbb{R}^3:\ \tfrac{1}{2}e^{l_{j-1}}\leq|x|\leq 3e^{l_{j-1}}\right\}$$

for $1\leq j\leq k-1$ we conclude, using again (2.8), by induction that

$$\begin{aligned}\operatorname{dens}\left(\overline{A}_{k+1},B\right)&=\operatorname{dens}\left((L_0\circ\dots\circ L_k)\left(\overline{U}(\ell_k)\right),(L_0\circ\dots\circ L_k)(f(R(r_k,\ell_k)))\right)\\&\geq\eta^{k+1}\delta.\end{aligned}\tag{4.3}$$

On the other hand, since $R(r_k,\ell_k)\subset H_{\geq\ell_k}$ we deduce from (2.9) that

$$\operatorname{diam}L_{k-1}(R(r_k,\ell_k))\leq\frac{c_4\pi}{\ell_k}\operatorname{diam}R(r_k,\ell_k)\leq\frac{3c_4\pi}{\ell_k}.$$

Putting $E_*(t)=\frac{1}{2}e^t$ and noting that $\ell_j\geq E_*(\ell_{j-1})$ we find that

$$\operatorname{diam}L_{k-1}(R(r_k,\ell_k))\leq\frac{3c_4\pi}{E_*^k(\ell_0)}.$$

Using (2.3) we deduce that

$$\operatorname{diam} B = \operatorname{diam}\left((L_0 \circ \ldots \circ L_{k-1})(R(r_k, \ell_k))\right) \leq \alpha^{k-1} \frac{3c_4\pi}{E_*^k(\ell_0)} \leq \frac{1}{2E_*^k(\ell_0)} \tag{4.4}$$

for large k.

Because of (4.3) and (4.4) we can apply Lemma 4.1 with

$$\Delta_k := \eta^{k+1}\delta \quad \text{and} \quad d_k := \frac{1}{2E_*^k(\ell_0)}.$$

Since $\eta < 1$ and $\delta < 1$ we have

$$\sum_{j=1}^{k} |\log \Delta_j| = \sum_{j=1}^{k} \left(\log \frac{1}{\delta} + (j+1)\log\frac{1}{\eta}\right) = k \log \frac{1}{\delta} + \frac{k^2 + 3k}{2} \log \frac{1}{\eta} \leq k^3$$

for large k. On the other hand,

$$|\log d_k| = \log\left(2E_*^k(\ell_0)\right) = E_*^{k-1}(\ell_0)$$

for large k. It is not difficult to see that

$$\lim_{k\to\infty} \frac{k^3}{E_*^{k-1}(\ell_0)} = 0.$$

Hence $\dim A = 3$ by Lemma 4.1. Since clearly $A \subset J$ we also have $\dim J = 3$.

Remark. It follows from the construction that $A \subset I(f)$. Thus we also have $\dim I(f) = 3$.

5. The Hausdorff dimension of the hairs without endpoints

In this section we prove that $J \backslash C$ has Hausdorff dimension 1, thereby establishing the analogue of Theorem C and thus completing the proof of Theorem 1.

The following lemma is standard [13, Lemma 4.8]. Here the open ball of radius r around a point $x \in \mathbb{R}^n$ is denoted by $B(x, r)$.

Lemma 5.1. *Let $K \subset \mathbb{R}^n$ be bounded, $R > 0$ and $r : K \to (0, R]$. Then there exists an at most countable subset L of K such that $B(x, r(x)) \cap B(y, r(y)) = \emptyset$ for $x, y \in L$, $x \neq y$, and*

$$\bigcup_{x\in K} B(x, r(x)) \subset \bigcup_{x\in L} B(x, 4r(x))$$

We shall deduce the following result from Lemma 5.1.

Lemma 5.2. *Let $K \subset \mathbb{R}^n$ be bounded and let $\rho > 1$. Suppose that for all $x \in K$ and $\delta > 0$ there exist $r(x) \in (0, 1)$, $d(x) \in (0, \delta)$ and $N(x) \in \mathbb{N}$ satisfying $N(x)d(x)^\rho \leq r(x)^n$ such that $B(x, r(x)) \cap K$ can be covered by $N(x)$ sets of diameter at most $d(x)$. Then* $\dim K \leq \rho$.

Proof. Choose $R > 0$ such that $K \subset B(0, R)$ and let $\delta > 0$. Since

$$K \subset \bigcup_{x\in K} B\left(x, \tfrac{1}{4}r(x)\right),$$

Lemma 5.1 yields the existence of an at most countable subset L of K such that

$$K \subset \bigcup_{x\in L} B(x, r(x))$$

while

$$B\left(x, \tfrac{1}{4}r(x)\right) \cap B\left(y, \tfrac{1}{4}r(y)\right) = \emptyset \quad \text{for } x, y \in L,\ x \neq y.$$

For each $x \in L$, let $A_1(x), A_2(x), \dots, A_{N(x)}(x)$ be the sets of diameter at most $d(x)$ which cover $B(x, r(x)) \cap K$ so that $N(x)d(x)^\rho \le r(x)^n$. Then

$$K \subset \bigcup_{x\in L} \bigcup_{j=1}^{N(x)} A_j(x).$$

Now

$$\sum_{x\in L} \sum_{j=1}^{N(x)} (\operatorname{diam} A_j(x))^\rho \le \sum_{x\in L} N(x)d(x)^\rho \le \sum_{x\in L} r(x)^n.$$

Since $r(x) \le \delta$ we have $B\left(x, \frac{1}{4}r(x)\right) \subset B\left(0, R + \frac{1}{4}\delta\right)$ for all $x \in L$. Since the balls $B\left(x, \frac{1}{4}r(x)\right)$, $x \in L$, are pairwise disjoint, this yields

$$\sum_{x\in L} \left(\tfrac{1}{4}r(x)\right)^n \le \left(R + \tfrac{1}{4}\delta\right)^n.$$

We obtain

$$\sum_{x\in L} \sum_{j=1}^{N(x)} (\operatorname{diam} A_j(x))^\rho \le (4R + \delta)^n.$$

Thus the ρ-dimensional Hausdorff measure of K is finite. In particular, $\dim K \le \rho$. □

As in Karpińska's paper [21] the key idea is to show that points in $J\backslash C$ escape to ∞ in a comparatively small region. To define such a region Ω, we consider the function

$$\psi : [1, \infty) \to [1, \infty),\ \psi(x) = \exp\left(\sqrt{\log x}\right),$$

and put

$$\Omega := \left\{(x_1, x_2, x_3) \in \mathbb{R}^3 :\ x_3 > \max\{1, M\} \text{ and } x_1^2 + x_2^2 < \psi(x_3)^2\right\}.$$

It is not difficult to see that

$$\lim_{x\to\infty} \frac{\psi(x)}{x^\varepsilon} = 0 \quad \text{for } \varepsilon > 0 \tag{5.1}$$

and that

$$\lim_{k\to\infty} \frac{E^k(t')}{\psi(E^k(t))} = 0 \quad \text{for } 0 < t' < t. \tag{5.2}$$

Lemma 5.3. *If $x \in J\backslash C$, then $f^k(x) \in \Omega$ for all large k.*

Proof. Let $x = g_{\underline{s}}(t)$ where $t > t_{\underline{s}}$. Since $f^k(x) = g_{\sigma^k(\underline{s})}(E^k(t))$ by (3.13) we have

$$|f^k(x) - (2s_{k,1}, 2s_{k,2}, E^k(t))| \le c_9 \tag{5.3}$$

by (3.14). Let now $t' \in (t_{\underline{s}}, t)$. Then

$$\lim_{k\to\infty} \frac{|s_k|}{E^k(t')} = 0$$

by (3.3). Combining this with (5.2) yields

$$\lim_{k\to\infty} \frac{|s_k|}{\psi\left(E^k(t)\right)} = 0. \tag{5.4}$$

The conclusion now follows from (5.3) and (5.4). □

We define

$$J' := \left\{x \in J\backslash C :\ f^k(x) \in \Omega \text{ for all } k \geq 0\right\}.$$

We shall show that

$$\dim J' = 1. \tag{5.5}$$

Since

$$J\backslash C = \bigcup_{k\geq 0} f^{-k}(J')$$

by Lemma 5.3, since f is locally bilipschitz, and since bilipschitz maps preserve Hausdorff dimension, it follows from (5.5) that $\dim(J\backslash C) = 1$ as claimed.

It remains to prove (5.5). In order to do so, we apply Lemma 5.2 to a bounded subset K of J', say $K = B(0,R) \cap J'$ where $R > 0$. Fix ρ and δ and let $x = g_{\underline{s}}(t) \in J'$. We want to show that there exist $r(x)$, $d(x)$ and $N(x)$ with the properties stated in the lemma.

For $k \in \mathbb{N}$ we put

$$B_k := B\left(f^k(x), \tfrac{1}{4}E^k(t)\right)$$

and note that (5.3) yields

$$\begin{aligned} J' \cap B_k &\subset \Omega \cap B_k \\ &\subset \left\{(y_1,y_2,y_3) \in \mathbb{R}^3 :\ \tfrac{1}{2}E^k(t) < y_3 < \tfrac{3}{2}E^k(t), y_1^2 + y_2^2 < \psi(y_3)^2\right\} \\ &\subset \left[-\psi\left(\tfrac{3}{2}E^k(t)\right), \psi\left(\tfrac{3}{2}E^k(t)\right)\right]^2 \times \left[\tfrac{1}{2}E^k(t), \tfrac{3}{2}E^k(t)\right] \end{aligned}$$

for large k. This implies that $J' \cap B_k$ can be covered by N_k cubes of side length 1, where

$$N_k \leq 5\psi\left(\tfrac{3}{2}E^k(t)\right)^2 E^k(t) \tag{5.6}$$

for large k. These cubes may be assumed to lie in $H_{\geq \frac{1}{2}E^k(t)}$. By (2.9) the preimage of such a cube under f has diameter at most d_k, where

$$d_k := \frac{2\sqrt{3}c_4\pi}{E^k(t)}. \tag{5.7}$$

Using (2.3) we see that the diameters of the preimage of these cubes under f^k also do not exceed d_k.

Let B_0 be the component of $f^{-k}(B_k)$ that contains x; that is,

$$B_0 = (L_0 \circ L_1 \circ \ldots \circ L_{k-1})(B_k)$$

with $L_j = \Lambda^{s_j}$ as in section 3. The above reasoning shows that $J' \cap B_0$ can be covered by N_k sets of diameter at most d_k, where N_k and d_k satisfy (5.6) and (5.7) if k is large.

In order to apply Lemma 5.2 we need to cover the intersection of J' with a ball around x and thus we denote by r_k the radius of the largest ball around x that is contained in B_0. We have to estimate r_k from below. In order to do so, let $y \in \partial B_0$ with $|y - x| = r_k$. Let γ_0 be the straight line segment connecting x and y, put $\gamma_j := f^j(\gamma_0)$ for $j = 1, 2, \ldots, k$ and $B_j := f^j(B_0)$ for $j = 1, 2, \ldots, k-1$. Then γ_k connects $f^k(x)$ to a point on ∂B_k and hence

$$\text{length}(\gamma_k) \geq \tfrac{1}{4}E^k(t). \tag{5.8}$$

It follows from (5.3) and the definition of Ω that

$$B_k \subset B\left(0, 2E^k(t)\right) \tag{5.9}$$

for large k. Since $\gamma_{k-1}=L_{k-1}(\gamma_k)$, we deduce from (2.6), (5.8) and (5.9) that

$$\text{length}(\gamma_{k-1}) \geq \frac{c_3}{2E^k(t)}\,\text{length}(\gamma_k) \geq \frac{c_3}{8}. \tag{5.10}$$

Also, (2.9) implies that

$$\operatorname{diam} B_{k-1} = \operatorname{diam} L_{k-1}(B_k) \leq \frac{2c_4\pi}{E^k(t)} \operatorname{diam} B_k = c_4\pi,$$

which together with (5.3) yields

$$\gamma_{k-1} \subset \overline{B_{k-1}} \subset B\left(0, 2E^{k-1}(t)\right)$$

if k is sufficiently large. Repeating the argument used to obtain (5.10) we find that

$$\text{length}(\gamma_{k-2}) \geq \frac{c_3}{2E^{k-1}(t)}\text{length}(\gamma_{k-1}).$$

Induction shows that

$$\text{length}(\gamma_{k-j}) \geq \left(\frac{c_3}{2}\right)^{j-1} \frac{1}{\prod_{l=1}^{j-1} E^{k-l}(t)} \frac{c_3}{8},$$

as long as $k-j+1$ is large enough to guarantee that

$$\overline{B_{k-j+1}} \subset B\left(0, 2E^{k-j+1}(t)\right).$$

We put $\tau := \frac{1}{2}c_3$ and deduce that there exist a positive constant κ such that

$$r_k = \text{length}(\gamma_0) \geq \kappa \frac{\tau^k}{\prod_{j=1}^{k-1} E^j(t)}. \tag{5.11}$$

For N_k, d_k and r_k satisfying (5.6), (5.7) and (5.11) and $\rho > 1$ we have

$$\frac{N_k d_k^\rho}{r_k^3} \leq \frac{5\psi\left(\frac{3}{2}E^k(t)\right)^2\left(2\sqrt{3}c_4\pi\right)^\rho}{E^k(t)^{\rho-1}\kappa^3\tau^{3k}} \left(\prod_{j=1}^{k-1} E^j(t)\right)^3. \tag{5.12}$$

Using (5.1) and the fact that

$$\lim_{k\to\infty} \frac{\prod_{j=1}^{k-1} E^j(t)}{E^k(t)^\varepsilon} = 0$$

for each $\varepsilon > 0$ it is not difficult to deduce that the right hand side of (5.12) tends to 0 as $k \to \infty$. In particular, we have $N_k d_k^\rho \leq r_k^3$ for large values of k. Moreover, $d_k \in (0,\delta)$ and $r_k \in (0,1)$ if k is large. We now take such a value of k and put $N(x) := N_k$, $d(x) := d_k$ and $r(x) := r_k$. Then the hypotheses of Lemma 5.2 are satisfied.

We conclude that $\dim(J' \cap B(0,R)) \leq \rho$. With $\rho \to 1$ and $R \to \infty$ we obtain (5.5). This completes the proof of Theorem 1.

6. Proof of Theorem 2

Let $x \in J$ and put $x_k = (x_{k,1}, x_{k,2}, x_{k,3}) := f^k(x_0)$ for $k \geq 0$. We shall recursively define a sequence $(y_k)_{k\geq 0}$ in $\mathbb{R}^3 \setminus J$ which has the following properties for certain positive constants η and μ:

- (i) $y_{k,3} = x_{k,3}$,
- (ii) $|y_k - x_k| \leq 4$,
- (iii) $f(y_k) \in H_{=M}$,
- (iv) $f(y_{k-1})$ and y_k can be connected by a curve $\gamma_k \subset \mathbb{R}^3 \setminus (J \cup B(0, \eta|x_k|))$ with $\text{length}(\gamma_k) \leq \mu|x_k|$, provided $k \geq 1$.

In order to do so we recall that $J \subset H_{>M}$. As before we put $\underline{s} = (s_k)_{k\geq 0} := \underline{s}(x)$ so that $x_k \in T(s_k)$ for all $k \geq 0$. In fact, we even have $x_k \in \Lambda^{s_k}(H_{>M}) \subset T(s_k)$ for $k \geq 0$. Thus we can choose $y_0 \in \Lambda^{s_0}(H_{=M}) \subset T(s_0)$ such that $y_{0,3} = x_{0,3}$. It follows that $|y_0 - x_0| \leq 4$ and $f(y_0) \in H_{=M}$. Thus (i), (ii) and (iii) are satisfied for $k = 0$.

Suppose now that $k \geq 1$ and that y_{k-1} has been defined. We put $z_k = (z_{k,1}, z_{k,2}, z_{k,3}) := f(y_{k-1})$ and note that $z_{k,3} = M$ by (iii) and

$$|x_k| - 2a \leq \exp(x_{k-1,3}) - a = \exp(y_{k-1,3}) - a \leq |z_k| \leq \exp(x_{k-1,3}) + a \leq |x_k| + 2a$$

by (2.5). Now there exists $r_k = (r_{k,1}, r_{k,2}) \in S$ such that $|(2r_{k,1}, 2r_{k,2}) - (z_{k,1}, z_{k,2})| \leq 4$. We put $u_k := (2r_{k,1} + 1, 2r_{k,2} + 1, M)$ and $v_k := (2r_{k,1} + 1, 2r_{k,2} + 1, x_{k,3})$. Since $x_k \in J \subset H_{>M}$ we have $|v_k| \geq |u_k| \geq \max\{M, |z_k| - 6\} \geq \max\{M, |x_k| - 6 - 2a\}$ and thus for small $\eta > 0$ the straight line segments $[z_k, u_k]$ and $[u_k, v_k]$ do not intersect $B(0, \eta|x_k|)$. Moreover, these line segments are contained in $\mathbb{R}^3 \setminus J$. We also have

$$\text{length}([z_k, u_k]) = |z_k - u_k| \leq 6 \tag{6.1}$$

while

$$\text{length}([u_k, v_k]) = x_{k,3} - M \leq |x_k|. \tag{6.2}$$

We define $w_k := (2s_{k,1} + 1, 2s_{k,2} + 1, x_{k,3})$. Then $w_k \notin J$,

$$|w_k - x_k| \leq 4 \tag{6.3}$$

and thus $|w_k| \geq \max\{M, |x_k| - 4\}$. It is not difficult to see that v_k and w_k can be connected by a curve

$$\sigma_k \subset H_{=x_{k,3}} \setminus (J \cup B(0, \min\{|v_k|, |w_k|\}))$$

which satisfies

$$\text{length}(\sigma_k) \leq 4\,(|v_k| + |w_k| + 4)\,.$$

The lower bounds for $|v_k|$ and $|w_k|$ obtained above show that σ_k does not intersect the ball $B(0, \eta|x_k|)$ if η is chosen small enough. We also have $|w_k| \leq |x_k| + 4$ and

$$|v_k| \leq |u_k| + x_{k,3} \leq |z_k| + 6 + |x_k| \leq 2|x_k| + 6 + 2a.$$

Thus

$$\text{length}(\sigma_k) \leq 4\,(3|x_k| + 14 + 2a) = 12|x_k| + 56 + 8a. \tag{6.4}$$

Now $x_k \in \Lambda^{s_k}(H_{>M})$ while $w_k \notin \Lambda^{s_k}(H_{>M})$. Thus $[w_k, x_k]$ intersects $\Lambda^{s_k}(H_{=M})$. We denote by y_k the point of intersection which is closest to w_k. Then y_k satisfies (i), (ii) and (iii). Clearly, the segment $[w_k, y_k]$ does not intersect J and for small η it also does not intersect $B(0, \eta|x_k|)$. The curve

$$\gamma_k := [z_k, u_k] \cup [u_k, v_k] \cup \sigma_k \cup [w_k, y_k]$$

then connects $z_k = f(y_{k-1})$ with y_k and (6.1), (6.2), (6.3) and (6.4) yield

$$\text{length}(\gamma_k) \leq 13|x_k| + 66 + 8a.$$

Since $x_k \geq M$ we deduce that $\text{length}(\gamma_k) \leq \mu|x_k|$ for some $\mu > 0$. Moreover, it follows from the definition of γ_k that $\gamma_k \in \mathbb{R}^3 \setminus (J \cup B(0, \eta|x_k|))$ for some $\eta > 0$. Thus (iv) holds.

Let now (y_k) and (γ_k) be the sequences constructed as above. We put

$$\Gamma_k := (L_0 \circ L_1 \circ \cdots \circ L_{k-1})(\gamma_k)$$

where $L_0, L_1, \ldots, L_{k-1}$ are as in section 3. From (2.7) we can deduce that

$$\text{length}(L_{k-1}(\gamma_k)) \leq c_4 \frac{\text{length}(\gamma_k)}{\eta|x_k|} \leq \frac{c_4 \mu}{\eta}. \tag{6.5}$$

Combining (6.3) and (6.5) with (2.3) yields

$$\operatorname{dist}(x,\Gamma_k) \le 4\alpha^k \quad \text{and} \quad \operatorname{length}(\Gamma_k) \le \frac{c_4\mu}{\eta}\alpha^{k-1}.$$

It follows that

$$\Gamma := \bigcup_{k=1}^{\infty} \Gamma_k \cup \{x\}$$

is a rectifiable curve with endpoints y_0 and x which except for the point x is contained in $\mathbb{R}^3 \setminus J$. Thus x is accessible from $\mathbb{R}^3 \setminus J$.

7. Examples of Zorich maps

We consider Zorich maps $F(x_1,x_2,x_3) = e^{x_3}h(x_1,x_2)$ for which there exists an annulus

$$A := \left\{(x_1,x_2) \in \mathbb{R}^2 : \ (s-\delta)^2 < x_1^2 + x_2^2 < (s+\delta)^2\right\}, \quad 0 < \delta < s < \tfrac{1}{4},$$

such that if $(x_1,x_2) = (r\cos\varphi, r\sin\varphi) \in A$, then

$$h(r\cos\varphi, r\sin\varphi) = \left(R(r)\cos\Phi(\varphi), R(r)\sin\Phi(\varphi), \sqrt{1-R(r)^2}\right),$$

with certain increasing and continuously differentiable functions $R : (s-\delta, s+\delta) \to (0,1)$ and $\Phi : \mathbb{R} \to \mathbb{R}$ such that $\Phi(\varphi + 2\pi) = \Phi(\varphi) + 2\pi$ for $\varphi \in \mathbb{R}$. We put $t := R(s)$, $w := \log(s/t)$ and $a := s\sqrt{1-t^2}/t - w$.

Then

$$f(x_1,x_2,x_3) = F(x_1,x_2,x_3) - (0,0,a) = e^{x_3}h(x_1,x_2) - (0,0,a)$$

maps the circle

$$C(s,w) := \left\{(x_1,x_2,x_3) \in \mathbb{R}^3 : \ x_1^2 + x_2^2 = s^2, \ x_3 = w\right\}$$

into itself. Now

$$\frac{\partial f}{\partial r}(r\cos\varphi, r\sin\varphi, x_3) = e^{x_3}\left(R'(r)\cos\Phi(\varphi), R'(r)\sin\Phi(\varphi), \frac{-R'(r)}{\sqrt{1-R(r)^2}}\right)$$

and hence

$$\left|\frac{\partial f}{\partial r}(s\cos\varphi, s\sin\varphi, w)\right| = e^w R'(s)\sqrt{1+\frac{1}{1-t^2}} = \frac{s}{t}R'(s)\sqrt{\frac{2-t^2}{1-t^2}}. \tag{7.1}$$

Also,

$$\frac{\partial f}{\partial x_3}(r\cos\varphi, r\sin\varphi, x_3) = e^{x_3}h(r\cos\varphi, r\sin\varphi)$$

so that

$$\left|\frac{\partial f}{\partial x_3}(s\cos\varphi, s\sin\varphi, w)\right| = e^w = \frac{s}{t}. \tag{7.2}$$

We can choose s and R such that $t = R(s) > 4s$ and

$$R'(s) < \frac{t}{4s}\sqrt{\frac{1-t^2}{2-t^2}}.$$

Then the right hand sides of both (7.1) and (7.2) are strictly less than $\frac{1}{4}$. This implies that there exists $\varepsilon > 0$ such that if $D := \{x \in \mathbb{R}^3 : \operatorname{dist}(x, C(s,w)) < \varepsilon\}$, then

$$\left|\frac{\partial f}{\partial r}(x)\right| \le \frac{1}{4} \quad \text{and} \quad \left|\frac{\partial f}{\partial x_3}(x)\right| \le \frac{1}{4}$$

for $x \in D$. It follows that if $x \in D$ and $y \in C(s,w)$ such that $|x-y| = \operatorname{dist}(x, C(s,w))$, then $|f(x)-f(y)| \leq \frac{1}{2}|x-y|$. Hence

$$\operatorname{dist}(f^k(x), C(s,w)) \to 0$$

as $k \to \infty$ for all $x \in D$.

The simplest choice for the function Φ is $\Phi(\varphi) = \varphi$. Then $C(s,w)$ is a continuum which consists of fixed points of f and attracts all points from a neighborhood of $C(s,w)$.

We may also choose a function Φ which satisfies

$$\Phi(\varphi) = \varphi + \varphi^3 \sin\left(\frac{\pi}{\varphi}\right) \quad \text{for } |\varphi| \leq \frac{1}{5}$$

since then

$$\Phi'(\varphi) = 1 + 3\varphi^2 \sin\left(\frac{\pi}{\varphi}\right) - \varphi\pi \cos\left(\frac{\pi}{\varphi}\right) \geq 1 - \frac{3}{25} - \frac{\pi}{5} > 0 \quad \text{for } |\varphi| \leq \frac{1}{5}.$$

With $\varphi_n := 1/n$ the points $u_n := (s\cos\varphi_n, s\sin\varphi_n, w)$ are fixed points of f for $n \geq 5$, and since

$$\Phi'(\varphi_n) = 1 - (-1)^n \frac{\pi}{n} \quad \text{for } n \geq 5$$

we see that u_n is an attracting fixed point of f if $n \geq 5$ is even; that is, there exists a neighborhood U_n of u_n such that $f^k(x) \to u_n$ as $k \to \infty$ for all $x \in U_n$. Thus f has infinitely many attracting fixed points. If $n \geq 5$ is odd, then φ_n is a repelling fixed point of Φ and thus u_n is a saddle point of f.

Quite generally, we can take a circle diffeomorphism Ψ and choose Φ such that the restriction of f to $C(s,w)$ is conjugate to Ψ.

We thus see that the dynamics of Zorich maps can be much more complicated than those of exponential maps. Presumably Zorich maps can also have “strange attractors”.

References

[1] K. Barański, Trees and hairs for some hyperbolic entire maps of finite order. Math. Z. 257 (2007), 33–59.

[2] K. Barański, Hausdorff dimension of hairs and ends for entire maps of finite order. Math. Proc. Cambridge Philos. Soc. 145 (2008), 719–737.

[3] K. Barański, B. Karpińska and A. Zdunik, Dimension properties of the boundaries of exponential basins. To appear in Bull. London Math. Soc., arXiv: 0902.1065v1.

[4] A. F. Beardon, Iteration of rational functions. Springer, New York, Berlin, Heidelberg 1991.

[5] W. Bergweiler, Iteration of meromorphic functions. Bull. Amer. Math. Soc. (N. S.) 29 (1993), 151–188.

[6] W. Bergweiler, A. Fletcher, J. Langley and J. Meyer, The escaping set of a quasiregular mapping. Proc. Amer. Math. Soc. 137 (2009), 641–651.

[7] R. L. Devaney, Complex exponential dynamics. To appear in Volume 3 of the Handbook of Dynamical Systems. Available at http://math.bu.edu/people/bob/papers.html

[8] R. L. Devaney and L. R. Goldberg, Uniformization of attracting basins for exponential maps. Duke Math. J. 55 (1987), 253–266.

[9] R. L. Devaney and M. Krych, Dynamics of exp(z). Ergodic Theory Dynam. Systems 4 (1984), 35–52.

[10] R. L. Devaney and F. Tangerman, Dynamics of entire functions near the essential singularity. Ergodic Theory Dynam. Systems 6 (1986), 489-503.

[11] A. E. Eremenko, On the iteration of entire functions. In “Dynamical systems and ergodic theory”, Banach Center Publications 23. Polish Scientific Publishers, Warsaw 1989, 339–345.

[12] A. E. Eremenko and M. Yu. Lyubich, Dynamical properties of some classes of entire functions. Ann. Inst. Fourier 42 (1992), 989–1020.

[13] K. Falconer, Fractal geometry. John Wiley & Sons, Chichester, 1990.

[14] A. Fletcher and D. A. Nicks, Quasiregular dynamics on the n-sphere, Preprint, arXiv: 0909.0217v1.

[15] L. Geyer, Quasikonforme Deformation in der Iterationstheorie. Diploma thesis, Technical University Berlin, 1994.
[16] J. Heinonen, The branch set of a quasiregular mapping. In "Proceedings of the International Congress of Mathematicians", Vol. II, Higher Ed. Press, Beijing, 2002, 691–700.
[17] A. Hinkkanen, Uniformly quasiregular semigroups in two dimensions. Ann. Acad. Sci. Fenn. Math. 21 (1996), 205–222.
[18] A. Hinkkanen, G. J. Martin and V. Mayer, Local dynamics of uniformly quasiregular mappings. Math. Scand. 95 (2004), 80–100.
[19] T. Iwaniec and G. Martin, Geometric function theory and non-linear analysis. Oxford Mathematical Monographs. Oxford University Press, New York, 2001.
[20] B. Karpińska, Area and Hausdorff dimension of the set of accessible points of the Julia sets of λe^z and $\lambda \sin z$. Fund. Math. 159 (1999), 269–287.
[21] B. Karpińska, Hausdorff dimension of the hairs without endpoints for $\lambda \exp z$. C. R. Acad. Sci. Paris Ser. I Math. 328 (1999), 1039–1044.
[22] M. Kisaka and M. Shishikura, On multiply connected wandering domains of entire functions. In "Transcendental dynamics and complex analysis", edited by P. J. Rippon and G. M. Stallard, LMS Lecture Note Series, Vol. 348. Cambridge University Press, Cambridge 2008, 217–250.
[23] G. J. Martin, Branch sets of uniformly quasiregular maps. Conform. Geom. Dyn. 1 (1997), 24–27.
[24] O. Martio and U. Srebro, Periodic quasimeromorphic mappings in R^n. J. Anal. Math. 28 (1975), 20-40.
[25] J. C. Mayer, An explosion point for the set of endpoints of the Julia set of $\lambda \exp(z)$. Ergodic Theory Dynam. Systems 10 (1990), 177–183.
[26] V. Mayer, Uniformly quasiregular mappings of Lattès type. Conform. Geom. Dyn. 1 (1997), 104–111.
[27] C. McMullen, Area and Hausdorff dimension of Julia sets of entire functions. Trans. Amer. Math. Soc. 300 (1987), 329–342.
[28] J. Milnor, Dynamics in one complex variable. Third edition. Ann. of Math. Studies 160. Princeton Univ. Press, Princeton, NJ, 2006.
[29] S. Morosawa, Y. Nishimura, M. Taniguchi and T. Ueda, Holomorphic dynamics. Cambridge Studies in Advanced Mathematics 66. Cambridge University Press, Cambridge, 2000.
[30] L. Rempe, Topological dynamics of exponential maps on their escaping sets. Ergodic Theory Dynam. Systems 26 (2006), 1939–1975.
[31] S. Rickman, Quasiregular mappings. Springer, Berlin, 1993.
[32] P. J. Rippon and G. M. Stallard, On questions of Fatou and Eremenko. Proc. Amer. Math. Soc. 133 (2005), 1119-1126.
[33] G. Rottenfußer, J. Rückert, L. Rempe and D. Schleicher, Dynamic rays of bounded-type entire functions. To appear in Ann. of Math., arXiv: 0704.3213v2.
[34] D. Schleicher, Attracting dynamics of exponential maps. Ann. Acad. Sci. Fenn. Math. 28 (2003), 3–34.
[35] D. Schleicher, The dynamical fine structure of iterated cosine maps and a dimension paradox. Duke Math. J. 136 (2007), 343–356.
[36] D. Schleicher and J. Zimmer, Escaping points of exponential maps. J. London Math. Soc. (2) 67 (2003), 380–400.
[37] H. Siebert, Fixed points and normal families of quasiregular mappings. J. Anal. Math. 98 (2006), 145–168.
[38] N. Steinmetz, Rational iteration. Walter de Gruyter, Berlin 1993.
[39] D. Sun and L. Yang, Quasirational dynamical systems (Chinese). Chinese Ann. Math. Ser. A 20 (1999), 673–684.
[40] D. Sun and L. Yang, Iteration of quasi-rational mapping. Progr. Natur. Sci. (English Ed.) 11 (2001), 16–25.
[41] V. A. Zorich, A theorem of M. A. Lavrent'ev on quasiconformal space maps. Math. USSR, Sb. 3 (1967), 389–403; translation from Mat. Sb. (N.S.) 74 (116), (1967), 417–433.

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