THE SIZE OF WIMAN-VALIRON DISKS

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Dedicated to Professor C.-C. Yang on the occasion of this 65th birthday

Abstract. Wiman-Valiron theory and results of Macintyre about "flat regions" describe the asymptotic behavior of entire functions in certain disks around points of maximum modulus. We estimate the size of these disks for Macintyre's theory from above and below.

1. Introduction

Let $f(z) = \sum_{n=0}^{\infty} a_n z^n$ be an entire function, $M(r, f) := \max_{|z|=r} |f(z)|$ its *maximum modulus* and $\mu(r, f) := \max_{n\geq 0} |a_n|r^n$ its *maximum term*. The largest n for which $\mu(r, f) = |a_n|r^n$ is denoted by $\nu(r, f)$ and called the *central index*. (Except for a discrete set of r-values there is only one integer n with $\mu(r, f) = |a_n|r^n$.) We say that a set $F \subset [1, \infty)$ has *finite logarithmic measure* if $\int_F dt/t < \infty$.

The main result of Wiman-Valiron theory says that there exists a set F of finite logarithmic measure such that if $|z_r| = r \notin F$, if $|f(z_r)| = M(r, f)$ and if z is sufficiently close to z_r, then

$$f(z) \sim \left(\frac{z}{z_r}\right)^{\nu(r,f)} f(z_r) \tag{1.1}$$

as $r \to \infty$. Equivalently,

$$f(e^{\tau} z_r) \sim e^{\nu(r,f)\tau} f(z_r)$$

if $|\tau|$ is sufficiently small. Wiman [20] obtained (1.1) for

$$|z| = r \quad \text{and} \quad |\arg z - \arg z_r| \leq \frac{1}{\nu(r, f)^{3/4+\delta}}$$

if $\delta > 0$ while Valiron [19, Theorem 29] proved (1.1) under the conditions

$$\big||z| - r\big| \leq \frac{Kr}{\nu(r, f)} \quad \text{and} \quad |\arg z - \arg z_r| \leq \frac{1}{\nu(r, f)^{15/16}},$$

for any given constant K. Macintyre [16] noted that (1.1) holds for

$$|z - z_r| \leq \frac{r}{\nu(r, f)^{1/2+\varepsilon}}$$

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if $\varepsilon > 0$. The sharpest estimates are due to Hayman [10] whose results imply that if

$$\psi(t) = t \cdot \log t \cdot \log\log t \cdot \ldots \cdot \log^{m-1} t \cdot (\log^m t)^{1+\varepsilon},$$

where $\varepsilon > 0$, $m \in \mathbb{N}$ and $\log^m$ denotes the m-th iterate of the logarithm, then (1.1) holds for

$$|z - z_r| \leq \frac{r}{\sqrt{\psi(\nu(r,f)) \log \psi(\nu(r,f))}}.$$

Results similar to those of Wiman-Valiron theory were obtained by Macintyre [16] with $\nu(r,f)$ replaced by

$$a(r,f) := \frac{d \log M(r,f)}{d \log r}.$$

Recall here that $\log M(r,f)$ is convex in $\log r$. Since convex functions have nondecreasing left and right derivatives and since they are differentiable except for an at most countable set, the derivative of $\log M(r,f)$ with respect to $\log r$ exists except possibly for a countable set of r-values. (Actually, by a result of Blumenthal (see [19, Section II.3]), the set of r-values where $\log M(r,f)$ is not differentiable is discrete.) To be definite, we shall always denote by $a(r,f)$ the right derivative of $\log M(r,f)$ with respect to $\log r$. Then $a(r,f)$ is nondecreasing and it can be shown that

$$a(r,f) = \frac{z_r f'(z_r)}{f(z_r)}$$

except for an at most countable set of r-values. The result of Macintyre [16, Theorem 3] says that

$$f(z) \sim \left(\frac{z}{z_r}\right)^{a(r,f)} f(z_r)$$

for

$$|z - z_r| \leq \frac{r}{(\log M(r,f))^{1/2+\varepsilon}} \tag{1.2}$$

as $r \to \infty$, $r \notin F$.

More recently, a result of this type was obtained in [2]. There it is not required that f is entire but only that f is as in the following definition.

Definition 1.1. Let D be an unbounded domain in $\mathbb{C}$ whose boundary consists of piecewise smooth curves. Suppose that the complement of D is unbounded. Let f be a complex-valued function whose domain of definition contains the closure $\overline{D}$ of D. Then D is called a *direct tract* of f if f is holomorphic in D and continuous in $\overline{D}$ and if there exists $R > 0$ such that $|f(z)| = R$ for $z \in \partial D$ while $|f(z)| > R$ for $z \in D$.

We note that every transcendental entire function has a direct tract. Let f, D, R be as in the above definition and put

$$M(r,f,D) := \max_{|z|=r, z \in D} |f(z)|.$$

Then $\log M(r,f,D)$ is again convex in $\log r$. Denoting by $a(r,f,D)$ the right derivative of $\log M(r,f,D)$ with respect to $\log r$ we see as before that $a(r,f,D)$ is nondecreasing and

$$a(r,f,D) = \frac{z_r f'(z_r)}{f(z_r)}$$

except for an at most countable set of r-values, with $z_r \in D$ such that $|z_r| = r$ and $|f(z_r)| = M(r,f,D)$. It follows from a result of Fuchs [7] that

$$\lim_{r\to\infty} \frac{\log M(r,f,D)}{\log r} = \infty \quad \text{and} \quad \lim_{r\to\infty} a(r,f,D) = \infty.$$

The main result of [2] says that if $\tau > \frac{1}{2}$, then there exists a set F of finite logarithmic measure such that

$$f(z) \sim \left(\frac{z}{z_r}\right)^{a(r,f,D)} f(z_r) \tag{1.3}$$

for

$$|z - z_r| < \frac{r}{a(r,f,D)^\tau} \tag{1.4}$$

as $r \to \infty$, $r \notin F$. In particular, the disk of radius $r/a(r,f,D)^\tau$ around z_r is contained in the direct tract D.

We investigate the question how large the disk around z_r in which (1.3) holds can be chosen. Our main result says that if $\psi : [t_0,\infty) \to (0,\infty)$ satisfies certain regularity conditions discussed below, then (1.3) holds for $|z - z_r| < r/\sqrt{\psi(a(r,f,D))}$ if

$$\int_{t_0}^{\infty} \frac{dt}{\psi(t)} < \infty \tag{1.5}$$

and if $r \notin F$ is sufficiently large, but (1.3) need not hold in this disk if

$$\int_{t_0}^{\infty} \frac{dt}{\psi(t)} = \infty. \tag{1.6}$$

The "interesting" functions for conditions (1.5) and (1.6) are functions like

$$\psi(t) = t\,(\log t)^\alpha$$

or, more generally,

$$\psi(t) = t \cdot \log t \cdot \log\log t \cdot \ldots \cdot \log^{m-1} t \cdot (\log^m t)^\alpha ,$$

where $\alpha > 0$ and $m \in \mathbb{N}$. Here (1.5) holds for $\alpha > 1$ while (1.6) holds for $\alpha \le 1$. For these functions we have

$$1 \le \frac{t\psi'(t)}{\psi(t)} \le 1 + o(1)$$

as $t \to \infty$. Therefore it does not seem to be a severe restriction to impose the condition that ψ is differentiable and satisfies

$$K \le \frac{t\psi'(t)}{\psi(t)} \le L \tag{1.7}$$

for certain constants K and L satisfying $0 \le K \le 1 < L$.

Our results are as follows.

Theorem 1.1. *Let $t_0 > 0$ and let $\psi : [t_0, \infty) \to (0, \infty)$ be a differentiable function satisfying* (1.5) *and* (1.7) *for some $K > 0$ and $L < 2$.*

Let f be a function with a direct tract D and let $z_r \in D$ with $|z_r| = r$ and $|f(z_r)| = M(r, f, D)$. Then there exists a set F of finite logarithmic measure such that

$$f(z) \sim \left(\frac{z}{z_r}\right)^{a(r,f,D)} f(z_r) \quad \text{for} \quad |z - z_r| \le \frac{r}{\sqrt{\psi(a(r, f, D))}} \tag{1.8}$$

as $r \to \infty$, $r \notin F$.

Theorem 1.2. *Let $t_0 > 0$ and let $\psi : [t_0, \infty) \to (0, \infty)$ be a differentiable function satisfying* (1.6) *and* (1.7) *for $K = 1$ and some $L < \frac{6}{5}$.*

Then there exists an entire function f which has exactly one tract D such that if r is sufficiently large and $|z| = r$, then the disk of radius $r/\sqrt{\psi(a(r, f, D))}$ around z contains a zero of f.

In particular it follows under the hypotheses of Theorem 1.2 that the disk mentioned is not contained in D and that (1.8) does not hold.

Remark 1. Our method also yields that if f is entire and z_r is a point of modulus r with $|f(z_r)| = M(r, f)$, then (1.8) holds with $a(r, f, D)$ replaced by $a(r, f)$. Here we only note that if D_r is the direct tract containing z_r, then

$$a(r, f) = a(r, f, D_r) = \frac{z_r f'(z_r)}{f(z_r)}$$

except for an at most countable set of r-values.

We also note that if ψ satisfies (1.5), then

$$a(r, f, D) \le \psi(\log M(r, f, D)) \tag{1.9}$$

outside a set of finite logarithmic measure. In fact, if $s_0 := \log M(r_0, f, D) \ge t_0$ and if F denotes the set of all $r \ge r_0$ where (1.9) does not hold, then

$$\int_F \frac{dt}{t} \le \int_F \frac{a(t, f, D)}{\psi(\log M(t, f, D))} \frac{dt}{t} \le \int_{r_0}^{\infty} \frac{a(t, f, D)}{\psi(\log M(t, f, D))} \frac{dt}{t} = \int_{s_0}^{\infty} \frac{dt}{\psi(t)} < \infty.$$

We deduce that the condition $|z - z_r| \le r/\sqrt{\psi(a(r, f, D))}$ in (1.8) can be replaced by

$$|z - z_r| \le \frac{r}{\sqrt{\psi(\psi(\log M(r, f, D)))}}.$$

For entire f we can again replace $M(r, f, D)$ by $M(r, f)$ if $|f(z_r)| = M(r, f)$. With $\psi(t) = t^{1+\delta}$ we recover Macintyre's condition (1.2).

Remark 2. In the papers on Wiman-Valiron theory cited above it is usually not required that $|f(z_r)| = M(r, f)$ but only that $|f(z_r)| \ge \eta M(r, f)$ for some $\eta \in (0, 1)$, possibly depending on r. It is then shown that (1.1) holds for z in some disk around z_r whose size depends on η. In [2] only the case $\eta = 1$ is considered, although the method allows to deal with the case $0 < \eta < 1$ as well. For the sake of simplicity we also restrict to the case $\eta = 1$ in this paper.

Remark 3. It was shown in [1] that the estimate on the size of the exceptional set F is best possible in Wiman-Valiron theory, and it follows from the results there that this also holds for Macintyre's theory and Theorem 1.1.

Remark 4. We do not discuss the numerous applications that the theories of Wiman-Valiron and Macintyre have found, but just mention some references with applications to complex differential equations [6, 12, 13, 21], distribution of zeros of derivatives [4, 14], and complex dynamics [2, 5, 11].

2. Proof of Theorem 1.1

Let D be a direct tract of f. The proof in [2] that (1.3) holds for z satisfying (1.4) relies on a lemma [2, Lemma 11.3] which says that if $\beta > \frac{1}{2}$, then there exists a set F of finite logarithmic measure such that

$$\log M(s, f, D) \leq \log M(r, f, D) + a(r, f, D) \log \frac{s}{r} + o(1) \tag{2.1}$$

for

$$\left|\log \frac{s}{r}\right| \leq \frac{1}{a(r, f, D)^{\beta}}, \tag{2.2}$$

uniformly as $r \to \infty$, $r \notin F$. In order to prove Theorem 1.1 we shall prove that if ψ satisfies the hypothesis of this theorem, then (2.2) can be replaced by

$$\left|\log \frac{s}{r}\right| \leq \frac{1}{\sqrt{\psi(a(r, f, D))}}. \tag{2.3}$$

In order to prove that (2.1) holds under the assumption (2.3) we use the following lemma.

Lemma 2.1. *Let $x_0 > 0$ and let $T : [x_0, \infty) \to (0, \infty)$ be nondecreasing. Let $t_0 := T(x_0)$ and let $\sigma_1, \sigma_2 : [t_0, \infty) \to (0, \infty)$ be nondecreasing functions such that*

$$\int_{t_0}^{\infty} \frac{dt}{\sigma_1(t)\sigma_2(t)} < \infty.$$

Suppose also that σ_2 is differentiable and satisfies

$$0 \leq \frac{t\sigma_2'(t)}{\sigma_2(t)} \leq 1 - \delta$$

for $t \geq t_0$ and some $\delta > 0$. Then there exists a set $E \subset [x_0, \infty)$ of finite measure such that if $x \notin E$, then

$$T\left(x + \frac{1}{\sigma_1(T(x))}\right) < T(x) + \sigma_2(T(x)) \tag{2.4}$$

and

$$T\left(x - \frac{1}{\sigma_1(T(x))}\right) > T(x) - \sigma_2(T(x)). \tag{2.5}$$

Proof. First we note that $x - 1/\sigma_1(T(x)) \geq x_0$ for sufficiently large x, say $x \geq x_0'$. Thus the left hand side of (2.5) is defined for $x \geq x_0'$. Denoting by E_1 the subset of $[x_0, \infty)$ where (2.4) fails and by E_2 the subset of $[x_0', \infty)$ where (2.5) fails we can thus take $E = [x_0, x_0'] \cup E_1 \cup E_2$.

We put $G(t) := t/\sigma_2(t)$. Since

$$\frac{tG'(t)}{G(t)} = 1 - \frac{t\sigma_2'(t)}{\sigma_2(t)} \geq \delta$$

the function G is increasing and hence

$$\begin{aligned} G\left(t + \sigma_2(t)\right) - G(t) &= \int_t^{t+\sigma_2(t)} G'(u)du \\ &\geq \delta \int_t^{t+\sigma_2(t)} \frac{G(u)}{u} du \\ &\geq \delta G(t) \int_t^{t+\sigma_2(t)} \frac{du}{u} \\ &= \delta G(t) \log\left(1 + \frac{1}{G(t)}\right) \end{aligned}$$

for $t \geq t_0$. Since the function $x \mapsto x \log(1 + 1/x)$ is increasing for $x > 0$ we deduce that

$$G\left(t + \sigma_2(t)\right) - G(t) \geq \eta := \delta G(t_0) \log\left(1 + \frac{1}{G(t_0)}\right) > 0 \tag{2.6}$$

for $t \geq t_0$. Similarly,

$$\begin{aligned} G(t) - G\left(t - \sigma_2(t)\right) &= \int_{t-\sigma_2(t)}^{t} G'(u)du \\ &\geq \delta \int_{t-\sigma_2(t)}^{t} \frac{G(u)}{u} du \\ &= \delta \int_{t-\sigma_2(t)}^{t} \frac{du}{\sigma_2(u)} \\ &\geq \delta \frac{1}{\sigma_2(t)} \int_{t-\sigma_2(t)}^{t} du \\ &= \delta \end{aligned} \tag{2.7}$$

for $t \geq t_0$.

To estimate the size of E_1 we may assume that E_1 is unbounded. We choose $x_1 \in E_1 \cap [\inf E_1, \inf E_1 + \frac{1}{2}]$ and put $x_1' := x_1 + 1/\sigma_1(T(x_1))$. Recursively we then choose

$$x_j \in E_1 \cap \left[\inf\left(E_1 \cap [x_{j-1}', \infty)\right), \inf\left(E_1 \cap [x_{j-1}', \infty)\right) + 2^{-j}\right]$$

and put $x_j' := x_j + 1/\sigma_1(T(x_j))$. Then

$$T(x_{j+1}) \geq T(x_j') = T\left(x_j + \frac{1}{\sigma_1(T(x_j))}\right) \geq T(x_j) + \sigma_2(T(x_j))$$

and hence

$$G(T(x_{j+1})) \geq G\left(T(x_j)+\sigma_2(T(x_j))\right) \geq G\left(T(x_j)\right)+\eta$$

by (2.6). Induction shows that

$$G(T(x_j)) \geq G\left(T(x_1)\right)+(j-1)\eta \tag{2.8}$$

for $j \in \mathbb{N}$. In particular it follows that $x_j \to \infty$ so that

$$E_1 \subset \bigcup_{j=1}^{\infty}\left[x_j-2^{-j}, x_j'\right].$$

Hence

$$\operatorname{meas} E_1 \leq \sum_{j=1}^{\infty}\left(x_j'-x_j+2^{-j}\right)=\sum_{j=1}^{\infty}\frac{1}{\sigma_1(T(x_j))}+1.$$

With $H := \sigma_1 \circ G^{-1}$ and $u_0 := G\left(T(x_1)\right)$ we deduce from (2.8) that

$$\sigma_1(T(x_j)) = H(G(T(x_j))) \geq H(u_0+(j-1)\eta).$$

Hence

$$\sum_{j=2}^{\infty}\frac{1}{\sigma_1(T(x_j))} \leq \sum_{j=2}^{\infty}\frac{1}{H(u_0+(j-1)\eta)} \leq \frac{1}{\eta}\int_{u_0}^{\infty}\frac{du}{H(u)} = \frac{1}{\eta}\int_{T(x_1)}^{\infty}\frac{G'(v)}{\sigma_1(v)}dv.$$

Since

$$G'(v) = \frac{1}{\sigma_2(v)} - \frac{v\sigma_2'(v)}{\sigma_2(v)^2} \leq \frac{1}{\sigma_2(v)}$$

we obtain

$$\sum_{j=2}^{\infty}\frac{1}{\sigma_1(T(x_j))} \leq \frac{1}{\eta}\int_{T(x_1)}^{\infty}\frac{dv}{\sigma_1(v)\sigma_2(v)} < \infty.$$

Altogether we have

$$\operatorname{meas} E_1 \leq \frac{1}{\sigma_1(t_0)} + \frac{1}{\eta}\int_{t_0}^{\infty}\frac{dv}{\sigma_1(v)\sigma_2(v)} + 1 < \infty.$$

To estimate E_2 we proceed similarly. We may assume that $E_2 \neq \emptyset$ and fix $R > x_0'$ so large that $E_2 \cap [x_0', R] \neq \emptyset$. We choose

$$z_1 \in E_2 \cap \left[\sup\left(E_2 \cap [x_0', R]\right) - \tfrac{1}{2}, \sup\left(E_2 \cap [x_0', R]\right)\right]$$

and put $z_1' := z_1 - 1/\sigma_1(T(z_1))$. Recursively we then choose

$$z_j \in E_2 \cap \left[\sup\left(E_2 \cap [x_0', z_{j-1}']\right) - 2^{-j}, \sup\left(E_2 \cap [x_0', z_{j-1}']\right)\right]$$

and put $z_j' := z_j - 1/\sigma_1(T(z_j))$, as long as $E_2 \cap [x_0', z_{j-1}'] \neq \emptyset$. However, since

$$\begin{aligned} T(z_{j+1}) &\leq T(z_j') \\ &= T\left(z_j - \frac{1}{\sigma_1(T(z_j))}\right) \\ &\leq T(z_j) - \sigma_2(T(z_j)) \\ &= \left(1 - \frac{1}{G(T(z_j))}\right) T(z_j) \\ &\leq \left(1 - \frac{1}{G(T(z_1))}\right) T(z_j) \\ &\leq \left(1 - \frac{1}{G(T(z_1))}\right)^j T(z_1), \end{aligned}$$

the process stops and we obtain two finite sequences $(z_1, \ldots, z_N)$ and $(z_1', \ldots, z_N')$ with

$$E_2 \cap [x_0', R] \subset \bigcup_{j=1}^{N} [z_j', z_j + 2^{-j}].$$

With $y_j := z_{N-j+1}$ we thus have

$$E_2 \cap [x_0', R] \subset \bigcup_{j=1}^{N} [y_j', y_j + 2^{j-N-1}]$$

and

$$T(y_j) \leq T(y_{j+1}) - \sigma_2(T(y_{j+1})).$$

Hence

$$G(T(y_j)) \leq G\left(T(y_{j+1}) - \sigma_2(T(y_{j+1}))\right) \leq G\left(T(y_{j+1})\right) - \delta$$

by (2.7) and thus

$$G(T(y_j)) \geq G(T(y_1)) + (j-1)\delta$$

by induction. Now the estimate for E_2 is very similar to that for E_1. We obtain

$$\begin{aligned} \text{meas}\,(E_2 \cap [x_0', R]) &\leq \sum_{j=1}^{N} (y_j - y_j' + 2^{j-N-1}) \\ &= \sum_{j=1}^{N} \frac{1}{\sigma_1(T(y_j))} + \sum_{j=1}^{N} 2^{j-N-1} \\ &\leq \frac{1}{\sigma_1(T(y_1))} + \frac{1}{\delta} \int_{T(y_1)}^{\infty} \frac{du}{H(u)} + 1 \\ &\leq \frac{1}{\sigma_1(t_0)} + \frac{1}{\delta} \int_{t_0}^{\infty} \frac{du}{\sigma_1(u)\sigma_2(u)} + 1 \end{aligned}$$

and hence meas $E_2 < \infty$. □

Remark. Lemma 2.1 was proved in [2, Lemma 11.1] in the case that $\sigma_1(t) = t^\beta$ and $\sigma_2(t) = t^{1-\alpha}$ where $0 < \alpha < \beta$. The method of proof used here is similar, going back to a classical lemma of Borel; see [3, §3.3], [8, p. 90] and [17].

Similarly as in [2] we apply Lemma 2.1 to the (right) derivative Φ' of a convex function Φ.

Lemma 2.2. *Let $x_0 > 0$ and let $\Phi : [x_0, \infty) \to (0, \infty)$ be increasing and convex. Let $t_0 := \Phi(x_0)$ and let $\psi : [t_0, \infty) \to (0, \infty)$ be a differentiable function satisfying* (1.5) *and* (1.7) *with $K > 0$ and $L < 2$. Then there exists a set $E \subset [x_0, \infty)$ of finite measure such that*

$$\Phi(x + h) \leq \Phi(x) + \Phi'(x)h + o(1) \quad \textit{for} \quad |h| \leq \frac{1}{\sqrt{\psi(\Phi'(x))}}, \quad x \notin E, \tag{2.9}$$

uniformly as $x \to \infty$.

Proof. First we note that $\lim_{x\to\infty} \Phi'(x)$ exists since Φ' is nondecreasing. It is easy to see that (2.9) holds without an exceptional set E if this limit is finite. Hence we assume that $\lim_{x\to\infty} \Phi'(x) = \infty$.

Let

$$V(t) := \int_t^\infty \frac{du}{\psi(u)}$$

so that $V'(t) = -1/\psi(t)$. We may assume that $K < 1$ and apply Lemma 2.1 with $T = \Phi'$ and

$$\sigma_1(t) = \sigma_2(t) = V(t)^{K/2}\sqrt{\psi(t)}. \tag{2.10}$$

To show that the hypotheses of this lemma are satisfied we note that

$$\int_{t_0}^t \frac{du}{\sigma_1(u)\sigma_2(u)} = \int_{t_0}^t \frac{V(u)^{-K}}{\psi(u)} du = \frac{1}{1-K}\left(V(t_0)^{1-K} - V(t)^{1-K}\right)$$

and thus

$$\int_{t_0}^\infty \frac{du}{\sigma_1(u)\sigma_2(u)} < \infty.$$

We also have

$$\frac{t\sigma_2'(t)}{\sigma_2(t)} = \frac{K}{2}\frac{tV'(t)}{V(t)} + \frac{1}{2}\frac{t\psi'(t)}{\psi(t)}. \tag{2.11}$$

Since $V'(t) = -1/\psi(t) < 0$ this implies that

$$\frac{t\sigma_2'(t)}{\sigma_2(t)} \leq \frac{1}{2}\frac{t\psi'(t)}{\psi(t)} \leq \frac{L}{2} < 1.$$

On the other hand, since ψ is increasing it follows from (1.5) that $\psi(t)/t \to \infty$ as $t \to \infty$ and thus we find, using (1.7), that

$$\begin{aligned} 0 &< -tV'(t) \\ &= \frac{t}{\psi(t)} \\ &= \int_t^\infty \left(\frac{u\psi'(u)}{\psi(u)^2} - \frac{1}{\psi(u)} \right) du \\ &= \int_t^\infty \left(\frac{u\psi'(u)}{\psi(u)} \right) \frac{du}{\psi(u)} - V(t) \\ &\leq (L-1)V(t). \end{aligned}$$

It follows that

$$\frac{tV'(t)}{V(t)} \geq -(L-1)$$

and this, together with (1.7) and (2.11), implies that

$$\frac{t\sigma_2'(t)}{\sigma_2(t)} \geq -\frac{K}{2}(L-1) + \frac{K}{2} = \frac{K(2-L)}{2} > 0.$$

Thus the hypotheses of Lemma 2.1 are satisfied.

Next we note that (2.10) yields that

$$\sigma_k(t) = o\left(\sqrt{\psi(t)}\right)$$

as $t \to \infty$ for $k \in \{1,2\}$. In particular, we find that $\sigma_k(t) \leq \sqrt{\psi(t)}$ for large t. Lemma 2.1 now yields that if $x \notin E$ is large and $0 < h \leq 1/\sqrt{\psi(\Phi'(x))}$, then

$$\begin{aligned} \Phi(x+h) &= \Phi(x) + \int_x^{x+h} \Phi'(u)du \\ &\leq \Phi(x) + \Phi'(x+h)h \\ &\leq \Phi(x) + \Phi'\left(x + \frac{1}{\sigma_1(\Phi'(x))}\right) h \\ &\leq \Phi(x) + \left(\Phi'(x) + \sigma_2(\Phi'(x))\right) h \\ &\leq \Phi(x) + \Phi'(x)h + \frac{\sigma_2(\Phi'(x))}{\sqrt{\psi(\Phi'(x))}} \end{aligned}$$

and hence $\Phi(x) + \Phi'(x)h + o(1)$ as $x \to \infty$. The case $-1/\sqrt{\psi(\Phi'(x))} \leq h < 0$ is analogous. □

Remark. If we apply Lemma 2.1 not to the functions defined by (2.10), as we did in the above proof, but to the functions $\sigma_1(t) = \sigma_2(t) = \sqrt{\psi(t)}$, then we obtain (2.9) with $o(1)$ replaced by 1. Choosing $\sigma_1(t) = \sigma_2(t) = \varepsilon\sqrt{\psi(t)}$ yields (2.9) with $o(1)$ replaced by ε.

We apply Lemma 2.2 to $\Phi(x) = \log M(e^x, f, D)$. Then $\Phi'(x) = a(e^x, f, D)$. With $r = e^x$ and $s = e^{x+h}$ we obtain

$$\begin{aligned}\log M(s, f, D) &= \Phi(x + h)\\ &\leq \Phi(x) + \Phi'(x)h + o(1)\\ &= \log M(r, f, D) + a(r, f, D) \log \frac{s}{r} + o(1)\end{aligned}$$

for $r \notin F = \exp E$, provided that

$$\left|\log \frac{s}{r}\right| = |h| \leq \frac{1}{\sqrt{\psi(\Phi'(x))}} = \frac{1}{\sqrt{\psi(a(r, f, D))}}.$$

This means that (2.1) holds for $r \notin F$ under the assumption (2.3).

The deduction of Theorem 1.1 from the result that (2.1) holds for s satisfying (2.3) if $r \notin F$ is similar to the arguments in [2] where the validity of (2.1) under the stronger condition (2.2) is used to show that (1.3) holds for z satisfying (1.4).

3. Proof of Theorem 1.2

3.1. **Preliminaries.** We first note that (1.6) and (1.7) also hold with $\psi(x)$ replaced by $\alpha\psi(\beta x)$ where $\alpha, \beta > 0$, and thus it suffices to show that there exist $\gamma, \delta > 0$ such that the disk of radius $\gamma r/\sqrt{\psi(\delta a(r, f, D))}$ around z contains a zero of f if $|z| = r$ is large. Moreover, we see that we may assume that $\psi(t_0) \geq t_0 \geq 1$.

We define $A_1 : [1, \infty) \to [t_0, \infty)$ by

$$\log r = \int_{t_0}^{A_1(r)} \frac{du}{\psi(u)}. \tag{3.1}$$

With $\phi : [t_0, \infty) \to [0, \infty)$,

$$\phi(t) := \int_{t_0}^{t} \frac{du}{\psi(u)}$$

we thus have $A_1(r) = \phi^{-1}(\log r)$. The function f constructed will satisfy

$$a(r, f) = a(r, f, D) \sim A_1(r)$$

as $r \to \infty$. However, before we can define the function f we will have to introduce some auxiliary functions and study their properties.

We first note that it follows from (1.7) and the assumption that $K = 1$ that

$$\log \frac{t}{t_0} \leq \log \frac{\psi(t)}{\psi(t_0)} \leq L \log \frac{t}{t_0}.$$

Using that $\psi(t_0) \geq t_0$ we see that

$$t \leq \psi(t) \leq ct^L \tag{3.2}$$

for $t \geq t_0$ and $c := \psi(t_0)t_0^{-L}$.

It follows from (3.1) that $A_1(r)$ is differentiable and $A_1'(r) = \psi(A_1(r))/r$. This implies that $A_2(r) := rA_1'(r) = \psi(A_1(r))$ is also differentiable so that we may define $A_3(r) := rA_2'(r)$. The functions A_1, A_2 and A_3 are thus related by

$$A_2(r) = \frac{dA_1(r)}{d\log r} = rA_1'(r) \quad \text{and} \quad A_3(r) = \frac{dA_2(r)}{d\log r} = rA_2'(r). \tag{3.3}$$

Since $\psi(t) \geq t$ we have $\phi(t) \leq \log(t/t_0)$ and thus $A_1(r) \geq t_0 r \geq r$ for $r \geq 1$. Using (3.2) and recalling that (1.7) holds with $K = 1$ we find that $A_2(r) \geq A_1(r)$ and

$$A_3(r) = rA_2'(r) = \psi'(A_1(r))A_2(r) \geq \frac{\psi(A_1(r))}{A_1(r)} A_2(r) \geq A_2(r).$$

Putting together the last estimates we thus have

$$A_3(r) \geq A_2(r) \geq A_1(r) \geq r \geq 1 > 0 \tag{3.4}$$

for $r \geq 1$. Combining this with (3.3) we see that A_1 and A_2 are increasing and that $A_1(r)$ is a convex function of $\log r$. Moreover, (1.7) yields that

$$1 \leq \frac{A_1(r)\psi'(A_1(r))}{\psi(A_1(r))} = \frac{A_1(r)A_3(r)}{A_2(r)^2} \leq L. \tag{3.5}$$

For $\rho > 1$ and $r > 1$ we thus have

$$\begin{aligned} \frac{1}{A_2(r)} - \frac{1}{A_2(\rho r)} &= \int_r^{\rho r} \frac{A_3(s)}{A_2(s)^2} \frac{ds}{s} \\ &\leq L \int_r^{\rho r} \frac{1}{A_1(s)} \frac{ds}{s} \\ &\leq \frac{L}{A_1(r)} \int_r^{\rho r} \frac{ds}{s} \\ &= \frac{L}{A_1(r)} \log \rho. \end{aligned}$$

Choosing

$$\rho := 1 + \frac{A_1(r)}{2A_2(r)} \tag{3.6}$$

we obtain

$$1 - \frac{A_2(r)}{A_2(\rho r)} \leq L \frac{A_2(r)}{A_1(r)} \log\left(1 + \frac{A_1(r)}{2A_2(r)}\right) \leq \frac{L}{2} \leq \frac{3}{5}$$

and hence

$$A_2\left(r\left(1 + \frac{A_1(r)}{2A_2(r)}\right)\right) = A_2(\rho r) \leq \frac{5}{2} A_2(r). \tag{3.7}$$

It follows from (3.2) that

$$A_2(r) = \psi(A_1(r)) \leq cA_1(r)^L \tag{3.8}$$

so that

$$A_1(r) \geq c^{-1/L} A_2(r)^{1/L}. \tag{3.9}$$

Together with (3.5) we deduce that

$$\begin{aligned} A_0(r) &:= \int_1^r A_1(s)\frac{ds}{s} \\ &\geq \frac{1}{L}\int_1^r \left(\frac{A_1(s)}{A_2(s)}\right)^2 A_3(s)\frac{ds}{s} \\ &\geq \frac{1}{Lc^{2/L}}\int_1^r A_2(s)^{2/L-2}A_3(s)\frac{ds}{s} \\ &= \frac{1}{c^{2/L}(2-L)}\left(A_2(r)^{2/L-1}-A_2(1)^{2/L-1}\right). \end{aligned}$$

Hence

$$A_2(r) = o\left(A_0(r)^{L/(2-L)}\right) \tag{3.10}$$

as $r\to\infty$. We also note that (3.5) yields

$$\begin{aligned} \frac{A_1(r)^2}{A_2(r)} &= \int_1^r A_1(s)\left(2-\frac{A_1(s)A_3(s)}{A_2(s)^2}\right)\frac{ds}{s}+\frac{A_1(1)^2}{A_2(1)} \\ &\geq (2-L)\int_1^r A_1(s)\frac{ds}{s} \\ &= (2-L)A_0(r) \end{aligned}$$

so that

$$\frac{A_0(r)A_2(r)}{A_1(r)^2} \leq \frac{1}{2-L} < \frac{5}{4}. \tag{3.11}$$

We now define $g:[1,\infty)\to[0,\infty)$,

$$g(r) := \int_1^r \sqrt{A_2(s)}\frac{ds}{s}$$

so that $g'(r)=\sqrt{A_2(r)}/r \geq 1/\sqrt{r}>0$. Thus g is increasing and hence the inverse function $h:=g^{-1}:[0,\infty)\to[1,\infty)$ exists. We will have to use various estimates involving the derivatives of h. First we note that

$$h'(t) = \frac{1}{g'(h(t))} = \frac{h(t)}{\sqrt{A_2(h(t))}}$$

and hence

$$\frac{h(t)}{h'(t)} = \sqrt{A_2(h(t))} \geq 1 \tag{3.12}$$

for $t\geq 0$ by (3.4). We deduce that

$$\frac{d}{dt}\left(\frac{h(t)}{h'(t)}\right) = \frac{A_2'(h(t))h'(t)}{2\sqrt{A_2(h(t))}} = \frac{A_3(h(t))h'(t)}{2\sqrt{A_2(h(t))}h(t)} = \frac{A_3(h(t))}{2A_2(h(t))}. \tag{3.13}$$

Similarly we find that

$$\frac{h''(t)}{h'(t)} = \left(1-\frac{A_3(h(t))}{2A_2(h(t))}\right)\frac{1}{\sqrt{A_2(h(t))}}$$

which together with (3.4), (3.5) and (3.8) yields that

$$\left|\frac{h''(t)}{h'(t)}\right| \leq \frac{3}{2}\frac{A_3(h(t))}{A_2(h(t))^{3/2}} \leq \frac{3L}{2}\frac{\sqrt{A_2(h(t))}}{A_1(h(t))} \leq \frac{3L\sqrt{c}}{2}A_1(h(t))^{L/2-1} = o(1)$$

as $t \to \infty$. It follows that if $0 \leq s \leq 1$, then

$$\log\frac{h'(t+s)}{h'(t)} = \int_t^{t+s}\frac{h''(u)}{h'(u)}du = o(1)$$

and hence

$$h'(t+s) \sim h'(t) \quad \text{for} \quad 0 \leq s \leq 1 \tag{3.14}$$

as $t \to \infty$. For later use we also note that (3.5), (3.13) and (3.9) yield that if $r > h(t)$, then

$$\begin{aligned}\left|\frac{d}{dt}\left(\frac{h(t)}{h'(t)}\log\frac{r}{h(t)}\right)\right| &= \left|\frac{A_3(h(t))}{2A_2(h(t))}\log\frac{r}{h(t)} - 1\right| \\ &\leq \frac{L}{2}\frac{A_2(h(t))}{A_1(h(t))}\log\frac{r}{h(t)} + 1 \\ &\leq \frac{Lc^{1/L}}{2}A_2(h(t))^{1-1/L}\log r + 1 \\ &\leq \frac{Lc^{1/L}}{2}A_2(r)^{1-1/L}\log r + 1.\end{aligned} \tag{3.15}$$

Finally we shall need the following two lemmas.

Lemma 3.1. *Let $R > 0$ and let $F : [0, R] \to \mathbb{R}$ be differentiable. Then*

$$\left|\sum_{k=1}^{[R]}F(k) - \int_{R-[R]}^{R}F(t)dt\right| \leq R\sup_{0<t<R}|F'(t)|.$$

Here $[R]$ denotes the integer part of R. The proof is straightforward and thus omitted. The following lemma is due to London [15, p. 502].

Lemma 3.2. *Let $\alpha, \beta : (0, \infty) \to (0, \infty)$ be functions such that α is convex, β is twice differentiable, β' is positive and unbounded and β'' is positive and continuous. Suppose that there exist $L > 0$ and $x_0 > 0$ such that*

$$\frac{\beta''(x)}{\beta'(x)} \leq L\frac{\beta'(x)}{\beta(x)}$$

for $x \geq x_0$. Suppose also that $\alpha(x) \sim \beta(x)$ as $x \to \infty$. Then $\alpha'(x) \sim \beta'(x)$ as $x \to \infty$.

3.2. **The maximum modulus of f.** Let h be as in the previous section. We define

$$f(z) := \prod_{k=1}^{\infty}\left(1 + \left(\frac{z}{h(k)}\right)^{\left[\frac{h(k)}{h'(k)}\right]}\right).$$

Note that $[h'(k)/h(k)] \geq 1$ for all $k \in \mathbb{N}$ by (3.12).

It will be apparent from the computations below that the infinite product converges absolutely and locally uniformly and thus defines an entire function which has $[h(k)/h'(k)]$ equally spaced zeros on the circle of radius $h(k)$ around 0. In this section we determine the asymptotic behavior of $\log M(r,f)$ and $a(r,f)$ as $r\to\infty$. In §3.3 we will then show that there exist $\gamma,\delta>0$ such that if $|z|$ is sufficiently large, then the disk of radius $\gamma|z|/\sqrt{\psi(\delta a(|z|,f))}$ contains a zero of f. Finally we will show in §3.4 that f has only one direct tract D so that $a(r,f)=a(r,f,D)$, thereby completing the proof of Theorem 1.2.

Let now $r>0$, define ρ by (3.6) and put

$$a_k := \log\left(1+\left(\frac{r}{h(k)}\right)^{\left[\frac{h(k)}{h'(k)}\right]}\right).$$

With

$$S_1 := \sum_{k=1}^{[g(r)]} a_k,\quad S_2 := \sum_{k=[g(r)]+1}^{[g(\rho r)]} a_k \quad\text{and}\quad S_3 := \sum_{k=[g(\rho r)]+1}^{\infty} a_k$$

we have

$$\log M(r,f)\le S_1+S_2+S_3.$$

First we note that

$$S_1 \le \sum_{k=1}^{[g(r)]}\left(\left[\frac{h(k)}{h'(k)}\right]\log\frac{r}{h(k)}+\log 2\right)\le\left(\sum_{k=1}^{[g(r)]}\frac{h(k)}{h'(k)}\log\frac{r}{h(k)}\right)+g(r)\log 2$$

and hence Lemma 3.1 and (3.15) yield that

$$S_1\le\int_0^{g(r)}\frac{h(t)}{h'(t)}\log\frac{r}{h(t)}dt+g(r)\left(\frac{Lc^{1/L}}{2}A_2(r)^{1-1/L}\log r+1\right)+g(r)\log 2.$$

Substitution and integration by parts yield

$$\begin{aligned}\int_0^{g(r)}\frac{h(t)}{h'(t)}\log\frac{r}{h(t)}dt &= \int_1^r sg'(s)^2\log\frac{r}{s}ds\\ &= \int_1^r\frac{A_2(s)}{s}\log\frac{r}{s}ds\\ &= \int_1^r A_1'(s)\log\frac{r}{s}ds \qquad (3.16)\\ &= \int_1^r A_1(s)\frac{ds}{s}-A_1(1)\log r\\ &= A_0(r)-t_0\log r.\end{aligned}$$

Moreover,

$$g(r)=\int_1^r\sqrt{A_2(s)}\frac{ds}{s}\le\sqrt{A_2(r)}\log r. \qquad (3.17)$$

Combining the above estimates we obtain

$$S_1\le A_0(r)+O\left(A_2(r)^{3/2-1/L}(\log r)^2\right)$$

as $r \to \infty$. Now (3.10) yields that

$$A_2(r)^{3/2-1/L} = A_2(r)^{(3L-2)/2L} = o\left(A_0(r)^{(3L-2)/(4-2L)}\right)$$

as $r \to \infty$. Since $L < \frac{6}{5}$ we have

$$\frac{3L-2}{4-2L} < 1.$$

Recalling that $A_2(r) \geq r$ we thus find that

$$A_2(r)^{3/2-1/L}(\log r)^2 = o\left(A_0(r)\right)$$

and hence that

$$S_1 \leq (1 + o(1))A_0(r)$$

as $r \to \infty$.

Next we note that $\rho < \frac{3}{2}$ by (3.4). Hence

$$\begin{aligned} S_2 &\leq g(\rho r)\log 2 \\ &\leq \sqrt{A_2(\rho r)}\log(\rho r)\log 2 \\ &\leq \sqrt{\frac{5}{2}A_2(r)}\left(\log r + \log\frac{3}{2}\right)\log 2 \\ &= O\left(A_0(r)^{L/(4-2L)}\log r\right) \\ &= o(A_0(r)) \end{aligned}$$

by (3.7), (3.10) and (3.17). Finally, using the abbreviation $\tau := \log\rho$ and noting that h/h' increases by (3.13), we have

$$\begin{aligned} S_3 &\leq \sum_{k=[g(\rho r)]+1}^{\infty} \left(\frac{r}{h(k)}\right)^{\left[\frac{h(k)}{h'(k)}\right]} \\ &\leq \sum_{k=[g(\rho r)]+1}^{\infty} \left(\frac{1}{\rho}\right)^{\frac{h(k)}{h'(k)}-1} \\ &= \rho \sum_{k=[g(\rho r)]+1}^{\infty} \exp\left(-\tau\frac{h(k)}{h'(k)}\right) \\ &\leq \rho\left(\int_{g(\rho r)}^{\infty} \exp\left(-\tau\frac{h(t)}{h'(t)}\right)dt + 1\right) \\ &= \rho\int_{\rho r}^{\infty} g'(s)\exp\left(-\tau s g'(s)\right)ds + \rho \\ &= \rho\int_{\rho r}^{\infty} \sqrt{A_2(s)}\exp\left(-\tau\sqrt{A_2(s)}\right)\frac{ds}{s} + \rho. \end{aligned}$$

Using (3.4) we thus find that

$$\begin{aligned} S_3 &\leq 2\rho \int_{\rho r}^{\infty} \frac{A_3(s)}{2\sqrt{A_2(s)}} \exp\left(-\tau\sqrt{A_2(s)}\right) \frac{ds}{s} + \rho \\ &= \frac{2\rho}{\tau} \exp\left(-\tau\sqrt{A_2(\rho r)}\right) + \rho \\ &\leq \frac{2\rho}{\tau} + \rho. \end{aligned}$$

Since $\rho < \frac{3}{2}$ and

$$\log(1+x) \geq x \log 2 \quad \text{for} \quad 1 \leq x \leq 2 \tag{3.18}$$

we have

$$\tau = \log \rho \geq (\rho - 1)\log 2 = \frac{A_1(r)}{2A_2(r)} \log 2 \geq \frac{\log 2}{2c^{1/L}} A_2(r)^{1/L-1}$$

by (3.9) and hence

$$S_3 \leq \frac{3}{\tau} + \frac{3}{2} \leq \frac{6c^{1/L}}{\log 2} A_2(r)^{1-1/L} + \frac{3}{2} = O\left(A_0(r)^{(L-1)/(2-L)}\right) = o(A_0(r))$$

by (3.10). Combining the estimates for S_1, S_2 and S_3 we conclude that

$$\log M(r, f) \leq (1 + o(1)A_0(r)$$

as $r \to \infty$.

On the other hand, denoting as usual (see [8, 9, 18]) by $N(r, 1/f)$ the counting function of the zeros of f, we have

$$\log M(r, f) \geq N\left(r, \frac{1}{f}\right) = \sum_{|c_j|<r} \log \frac{r}{|c_j|}$$

where $c_1, c_2, \ldots$ are the zeros of f. We obtain

$$N\left(r, \frac{1}{f}\right) = \sum_{k=1}^{[g(r)]} \left[\frac{h(k)}{h'(k)}\right] \log \frac{r}{h(k)}$$

and we see as in the estimate for S_1 that

$$N\left(r, \frac{1}{f}\right) \geq \int_0^{g(r)} \frac{h(t)}{h'(t)} \log \frac{r}{h(t)} dt - o(A_0(r)) = (1 - o(1))A_0(r).$$

Altogether we thus have

$$\log M(r, f) \sim A_0(r) \tag{3.19}$$

as $r \to \infty$. It follows from (3.11) and Lemma 3.2, applied to $\alpha(x) = \log M(e^x, f)$ and $\beta(x) = A_0(e^x)$, that

$$a(r, f) \sim A_1(r) \tag{3.20}$$

as $r \to \infty$.

3.3. **The distance to the closest zero.** For $z \in \mathbb{C}$ we denote by $\delta(z)$ the distance of z to the closest zero of f and we put $d(r) := \max_{|z|=r} \delta(z)$ for $r > 0$. For $r > h(1)$ we put $n := [g(r)]$ so that $n \geq 1$ and $h(n) \leq r \leq h(n+1)$. As f has $[h(n)/h'(n)]$ equally spaced zeros on the circle with radius $h(n)$ it follows that

$$d(r) \leq r - h(n) + \frac{2\pi h(n)}{\left[\frac{h(n)}{h'(n)}\right]} \leq h(n+1) - h(n) + 7h'(n)$$

for large r. By (3.14) we have $h'(n) \sim h'(g(r))$ and

$$h(n+1) - h(n) = \int_n^{n+1} h'(u)du \sim h'(g(r))$$

as $r \to \infty$. Together with (3.20) we thus find that

$$d(r) \leq 9h'(g(r)) = \frac{9}{g'(r)} = \frac{9r}{\sqrt{A_2(r)}} = \frac{9r}{\sqrt{\psi(A_1(r))}} \leq \frac{9r}{\sqrt{\psi\left(\frac{1}{2}a(r,f)\right)}}$$

for large r. As mentioned at the beginning of the proof, the method thus also yields a function f with $d(r) \leq r/\sqrt{\psi\left(a(r,f)\right)}$ for large r.

3.4. **The minimum modulus of f.** For $|z| = r_n := h\left(n + \frac{1}{2}\right)$ where $n \in \mathbb{N}$ we have

$$\log|f(z)| \geq \sum_{k=1}^{n} \log(b_k - 1) - \sum_{k=n+1}^{\infty} \log\left(1 + b_k\right) \tag{3.21}$$

where

$$b_k := \left(\frac{r_n}{h(k)}\right)^{\left[\frac{h(k)}{h'(k)}\right]}.$$

Noting that $[g(r_n)] = n$ we see that the estimates for S_2 and S_3 in §3.2 show that

$$\sum_{k=n+1}^{\infty} \log\left(1 + b_k\right) = o(A_0(r_n)) \tag{3.22}$$

as $n \to \infty$. To estimate the first sum on the right hand side of (3.21) we note that if $r_n \geq 2h(k)$, then $b_k \geq 2$. On the other hand, using (3.18) we see that if

$r_n < 2h(k)$, then

$$\begin{aligned}
\log b_k &= \left[\frac{h(k)}{h'(k)}\right] \log\left(\frac{r_n}{h(k}\right) \\
&= \left[\frac{h(k)}{h'(k)}\right] \log\left(1 + \frac{h\left(n+\frac{1}{2}\right) - h(k)}{h(k)}\right) \\
&\geq \log 2 \left[\frac{h(k)}{h'(k)}\right] \frac{h\left(n+\frac{1}{2}\right) - h(k)}{h(k)} \\
&\geq \frac{1}{2} \frac{h\left(n+\frac{1}{2}\right) - h(k)}{h'(k)} \\
&= \frac{1}{2} \frac{1}{h'(k)} \int_k^{k+\frac{1}{2}} h'(t)dt
\end{aligned}$$

for large n. Using (3.14) we see that $\log b_k \geq \frac{1}{5}$ for these values of k, provided n is sufficiently large. Since $2 \geq \exp\frac{1}{5}$ we thus have $b_k \geq \exp\frac{1}{5}$ for all $k \leq n$ if n is large. With $B := \frac{1}{5} - \log\left(\exp\frac{1}{5} - 1\right)$ we have

$$\log(b-1) \geq \log(b) - B \quad \text{for} \quad b \geq \exp\frac{1}{5}$$

and thus

$$\sum_{k=1}^{n} \log(b_k - 1) \geq \sum_{k=1}^{n} \log b_k - nB = \sum_{k=1}^{n} \left[\frac{h(k)}{h'(k)}\right] \log\left(\frac{r_n}{h(k)}\right) - nB$$

for large n. Using Lemma 3.1 and (3.16) we conclude as in §3.2 that

$$\sum_{k=1}^{n} \log(b_k - 1) \geq (1 - o(1))A_0(r_n).$$

Combining this with (3.22) this yields

$$\min_{|z|=r_n} \log|f(z)| \geq (1 - o(1))A_0(r_n).$$

In particular, $\min_{|z|=r_n} \log|f(z)| \to \infty$ as $n \to \infty$. It follows that f has exactly one direct tract. This completes the proof of Theorem 1.2.

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