NEWTON'S METHOD AND BAKER DOMAINS

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Abstract. We show that there exists an entire function f without zeros for which the associated Newton function $N_f(z) = z - f(z)/f'(z)$ is a transcendental meromorphic functions without Baker domains. We also show that there exists an entire function f with exactly one zero for which the complement of the immediate attracting basin has at least two components and contains no invariant Baker domains of N_f. The second result answers a question of J. Rückert and D. Schleicher while the first one gives a partial answer to a question of X. Buff.

1. Introduction and results

Newton's method for finding the zeros of an entire f consists of iterating the meromorphic function

$$N_f(z) := z - \frac{f(z)}{f'(z)},$$

see [1] for an introduction to the iteration theory of meromorphic functions, including a section on Newton's method. If ξ is a zero of f, then there exists an N_f-invariant component U of the Fatou set of N_f containing ξ in which the iterates N_f^k of N_f converge to ξ as $k \to \infty$. This domain U is called the *immediate basin* of ξ.

There may also be N_f-invariant components of the Fatou set of N_f in which the iterates of N_f tend to ∞. We call such an N_f-invariant domain a *virtual immediate basin*. (This is in slight deviation from [4, 10] where the definition of a virtual immediate basin additionally includes the existence of an "absorbing set"; cf. the remark in §3.3.) It was suggested by Douady that the existence of virtual immediate basins is related to 0 being an asymptotic value of f. This relationship was investigated in [2, 4]. While it was shown in [2] that in general the existence of a virtual immediate basin does not imply that 0 is an asymptotic value of f, this conclusion was shown to be true under suitable additional hypotheses in [4]. It was also shown in [4] that if f has a logarithmic singularity over 0, then N_f has a virtual immediate basin.

If f has the form $f = Pe^Q$ where P and Q are polynomials, with Q nonconstant, then the Newton function N_f is rational, ∞ is a parabolic fixed point of N_f and the associated parabolic domains are virtual immediate basins. If f does not have the above form, then N_f is transcendental. An invariant Fatou component where the iterates of N_f tend to ∞ is then called an *invariant Baker domain.* So except in the case where $f = Pe^Q$ a virtual immediate basin is an invariant Baker domain.

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If f has no zeros, then f has the asymptotic value 0 by Iversen's theorem [8, p. 292]. This suggests that there could always be virtual immediate basins if there are no zeros. We show that this is not the case in general.

Theorem 1. *There exists an entire function f without zeros for which N_f is a transcendental meromorphic function without invariant Baker domains.*

The following corollary is obvious.

Corollary. *There exists a transcendental meromorphic function without fixed points and without invariant Baker domains.*

This is a partial answer to a question of Buff who had asked whether there exists a transcendental *entire* function without fixed points and without invariant Baker domains.

Rückert and Schleicher [10] have shown that if f is a polynomial and if U is the immediate basin of a zero, then each component of $\mathbb{C} \setminus U$ contains the basin of another zero. They deduce this result from a more general result dealing with the case that f is entire but not necessarily a polynomial. To state this result, let again U be the immediate basin of a zero ξ of f and suppose that there are two N_f-invariant curves Γ_1 and Γ_2 which connect ξ to ∞ in $U \cup \{\infty\}$, which intersect only in ξ and ∞ and which are not homotopic (with fixed endpoints) in $U \cup \{\infty\}$. Let $\widetilde{V}$ be a component of $\mathbb{C} \setminus (\Gamma_1 \cup \Gamma_2)$. With these notations their main result [10, Theorem 5.1] takes the following form.

Theorem (Rückert, Schleicher). *If no point in $\widehat{\mathbb{C}}$ has infinitely many preimages within $\widetilde{V}$, then the set $V := \widetilde{V} \setminus U$ contains an immediate basin or a virtual immediate basin of N_f.*

Rückert and Schleicher raise the question whether the hypothesis that no point in $\widehat{\mathbb{C}}$ has infinitely many preimages within $\widetilde{V}$ is necessary. We show that this is indeed the case.

Theorem 2. *There exists an entire function g with exactly one zero at 0 such that the immediate basin of 0 contains $\mathbb{R}$, but N_g has no virtual immediate basin.*

The functions f and g in Theorems 1 and 2 can be given explicitly. Let (r_k) be a sequence of real numbers tending to ∞ and let (n_k) be a sequence of positive integers satisfying $n_k \geq k$ for all $k \in \mathbb{N}$. Then

$$h(z) := \prod_{k=1}^{\infty} \left(1 + \left(\frac{z}{r_k} \right)^{n_k} \right) \tag{1.1}$$

defines an entire function h. We shall show that if

$$r_k \geq 2r_{k-1} \geq 2, \quad n_k \geq \sum_{j=1}^{k-1} n_j \quad \text{and} \quad n_k \geq r_k^{4n_{k-1}} \tag{1.2}$$

for $k \geq 2$, then the functions

$$f(z) := \exp\left(\int_0^z h(t)dt \right) \tag{1.3}$$

and

$$g(z) := z \exp\left(\int_0^z \frac{h(t)-1}{t}dt\right) \tag{1.4}$$

satisfy the conclusions of Theorems 1 and 2, respectively.

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2. Proofs of Theorem 1 and 2

We denote the open disk of radius r around a point $a \in \mathbb{C}$ by $D(a,r)$. The hyperbolic metric in a plane domain U is denoted by λ_U. By $\text{dist}_U(z,A)$ we denote the hyperbolic distance between a point z and a set A. We shall make use of the fact that if $A \subset U$ and if z_0 is a point which is in the boundary of U but not in the closure of A, then $\text{dist}_U(z,A) \to \infty$ as $z \to z_0$, $z \in U$. In particular, we have

$$\lim_{z\to\infty} \text{dist}_{\mathbb{C}\setminus\{0,1\}}\left(z, D\left(\tfrac{1}{2},\tfrac{1}{2}\right)\right) = \infty. \tag{2.1}$$

To estimate the growth of h we note that (1.2) implies that if $k \geq 3$, then $r_k \geq 2^{k-1} \geq 4$ and $n_{k-1} \geq r_{k-1}^{4n_{k-2}} \geq 2^{(k-1)4n_{k-2}} \geq k+1$. For $|z| = r_k$ where $k \geq 3$ we thus obtain

$$\begin{aligned}
\log|h(z)| &\leq \sum_{j=1}^{k-1} \log\left(1 + \left|\frac{r_k}{r_j}\right|^{n_j}\right) + \log 2 + \sum_{j=k+1}^{\infty} \log\left(1 + \left|\frac{r_k}{r_j}\right|^{n_j}\right) \\
&\leq \sum_{j=1}^{k-1} \log\left(1 + {r_k}^{n_j}\right) + \log 2 + \sum_{j=k+1}^{\infty} \left|\frac{r_k}{r_j}\right|^{n_j} \\
&\leq \sum_{j=1}^{k-1} \log\left(2{r_k}^{n_j}\right) + \log 2 + \sum_{j=k+1}^{\infty} 2^{-n_j} \\
&\leq k\log 2 + \left(\sum_{j=1}^{k-1} n_j\right)\log r_k + 1 \\
&\leq (k + 2n_{k-1} + 1)\log r_k \\
&\leq 3n_{k-1}\log r_k.
\end{aligned}$$

Hence

$$|h(z)| \leq r_k^{3n_{k-1}}. \tag{2.2}$$

for $|z| = r_k$ and $k \geq 3$.

Proof of Theorem 1. Let h and f be defined by (1.1) and (1.3) so that

$$N_f(z) = z - \frac{1}{h(z)}.$$

Suppose that N_f has an invariant Baker domain U. Take $z_0 \in U$ and connect z_0 by a curve γ_0 in U to $N_f(z_0)$. Then $\gamma := \bigcup_{j=0}^{\infty} N_f^j(\gamma_0)$ is a curve in U which connects z_0 to ∞. By compactness, there exists $K \geq 0$ such that $\lambda_U(z, N_f(z)) \leq K$ for all

$z \in \gamma_0$. Since every $z \in \gamma$ has the form $z = N_f^j(\zeta)$ for some $\zeta \in \gamma_0$ and some $j \geq 0$ and since the holomorphic self-map N_f of U does not increase hyperbolic distances this implies that

$$\lambda_U(z, N_f(z)) \leq K \quad \text{for} \quad z \in \gamma. \tag{2.3}$$

For large k the curve γ intersects the circle $\{z : |z| = r_k\}$. Let z_k be a point of intersection. Define

$$P_k := \left\{ r_k e^{(2\nu+1)\pi i/n_k} : 0 \leq \nu \leq n_k - 1 \right\}.$$

The n_k points of P_k are zeros of h and hence poles of N_f. Thus $P_k \cap U = \emptyset$ for all $k \in \mathbb{N}$. For $k \geq 2$ we have $n_k \geq r_k^4 \geq 16$ so that P_k contains more than one point. Let a_k, b_k the points of P_k which are closest to z_k. Then

$$|a_k - b_k| = \left| e^{2\pi i/n_k} - 1 \right| \leq \frac{4\pi}{n_k} \tag{2.4}$$

and

$$z_k \in D\left(\tfrac{1}{2}(a_k + b_k), \tfrac{1}{2}|a_k - b_k|\right). \tag{2.5}$$

Define $L_k : \mathbb{C} \setminus \{a_k, b_k\} \to \mathbb{C} \setminus \{0, 1\}$ by $L_k(z) = (z - a_k)/(b_k - a_k)$. Then

$$\lambda_{\mathbb{C}\setminus\{0,1\}}(L_k(z_k), L_k(N_f(z_k))) = \lambda_{\mathbb{C}\setminus\{a_k,b_k\}}(z_k, N_f(z_k)) \leq \lambda_U(z_k, N_f(z_k)) \leq K \tag{2.6}$$

by (2.3). By (2.5) we have $L_k(z_k) \in D\left(\frac{1}{2}, \frac{1}{2}\right)$. On the other hand, (2.2), (2.4) and (1.2) imply that

$$\begin{aligned} |L_k(N_f(z_k))| &\geq |L_k(N_f(z_k)) - L_k(z_k)| - |L_k(z_k)| \\ &= \frac{|N_f(z_k) - z_k|}{|a_k - b_k|} - |L_k(z_k)| \\ &= \frac{1}{|h(z_k)(a_k - b_k)|} - |L_k(z_k)| \\ &\geq \frac{n_k}{4\pi r_k^{3n_{k-1}}} - 1 \\ &\geq \frac{r_k^{n_{k-1}}}{4\pi} - 1 \end{aligned}$$

and thus $|L_k(N_f(z_k))| \to \infty$ as $k \to \infty$. Combining this with (2.1) we see that $\lambda_{\mathbb{C}\setminus\{0,1\}}(L_k(z_k), L_k(N_f(z_k))) \to \infty$ as $k \to \infty$, contradicting (2.6). □

Proof of Theorem 2. Let h and g be defined by (1.1) and (1.4) so that

$$N_g(z) = z - \frac{z}{h(z)} = z\left(1 - \frac{1}{h(z)}\right).$$

The proof that N_g has no Baker domains proceeds exactly as the proof of Theorem 1. (We only obtain

$$|L_k(N_f(z_k))| \geq \frac{r_k^{n_{k-1}-1}}{4\pi} - 1,$$

but this still gives a contradiction.)

Since h is real on the real axis, the same holds for N_g, and since $h(x) > 1$ for all $x \in \mathbb{R} \setminus \{0\}$ we see that $|N_g(x)| < |x|$ for all $x \in \mathbb{R} \setminus \{0\}$. This implies that

$N_g^k(x) \to 0$ as $k \to \infty$, for all $x \in \mathbb{R}$. Hence $\mathbb{R}$ is contained in the immediate basin of 0. □

3. Remarks

1. It follows from the result of Buff and Rückert [4] already mentioned in the introduction that the function f of Theorem 1 has no logarithmic singularity over 0. Another example of an entire function without zeros and with no logarithmic singularity over 0 was given in [3].

2. The invariant components of the Fatou set of a meromorphic function can be classified; see [1]. For functions without fixed points there are only two possible types of invariant components: Baker domains and Herman rings. Fagella, Jarque and Taixes [6] have shown that a meromorphic function without fixed points does not have Herman rings. This implies that for a function f satisfying the conclusion of Theorem 1 the Fatou set of N_f has no invariant component at all. Probably there also exist entire functions f for which the Fatou set of N_f is empty.

3. It was shown by Przyztycki [9] that if f is a poynomial, then the immediate basin of each zero is simply connected. Shishikura [11] showed that in fact the Julia set of N_f is connected; that is, all Fatou components of N_f are simply connected. It is not known whether this last result also holds if f is an entire transcendental function, but Mayer and Schleicher [7] have shown that immediate basins are simply connected. Fagella, Jarque and Taixes [5, 6] have extended this result by showing that immediate attracting and parabolic basins (of any period) are simply connected and that preimages of simply connected Fatou components of N_f are simply connected. However, it remains open whether invariant Baker domains of N_f are necessarily simply connected. If this is true, then our definition of virtual immediate basins coincides with that given in [4, 10] since then the additional condition on the existence of an absorbing set is always satisfied; cf. the discussion in [4, 10].

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