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DYNAMICS OF MEROMORPHIC FUNCTIONS WITH DIRECT OR LOGARITHMIC SINGULARITIES

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Abstract. Let f be a transcendental meromorphic function and denote by $J(f)$ the Julia set and by $I(f)$ the escaping set. We show that if f has a direct singularity over infinity, then $I(f)$ has an unbounded component and $I(f) \cap J(f)$ contains continua. Moreover, under this hypothesis $I(f) \cap J(f)$ has an unbounded component if and only if f has no Baker wandering domain. If f has a logarithmic singularity over infinity, then the upper box dimension of $I(f) \cap J(f)$ is 2 and the Hausdorff dimension of $J(f)$ is strictly greater than 1. The above theorems are deduced from more general results concerning functions which have "direct or logarithmic tracts", but which need not be meromorphic in the plane. These results are obtained by using a generalization of Wiman-Valiron theory. This method is also applied to complex differential equations.

1. Introduction

For a function f meromorphic in the plane the *Fatou set* $F(f)$ is defined as the set where the iterates f^n of f are defined and form a normal family, and the *Julia set* $J(f)$ is its complement. The *escaping set* $I(f)$ is defined as the set of all $z \in \mathbb{C}$ for which $(f^n(z))$ is defined and $f^n(z) \to \infty$ as $n \to \infty$.

While the main objects studied in complex dynamics are the Fatou and Julia sets, the escaping set also plays a major role in the iteration theory of entire functions, beginning with a paper by Eremenko [12] who proved that if f is an entire transcendental function, then $I(f) \neq \emptyset$, $\partial I(f) = J(f)$, $I(f) \cap J(f) \neq \emptyset$ and $\overline{I(f)}$ has no bounded components. Eremenko conjectured that in fact all components of $I(f)$ are unbounded. While this conjecture is still open, it is known that $I(f)$ has at least one unbounded component [36]. A detailed study of the escaping set of an exponential function was given by Schleicher and Zimmer [43]. In particular, they confirmed Eremenko's conjecture for such functions. Since then the conjecture has also been proved for certain more general classes of entire functions by Barański [2], Rempe [30] and Rottenfußer, Rückert, Rempe and Schleicher [41]. Actually it was shown in [2, 41, 43] that – in the classes of functions considered – every point in $I(f)$ can be connected to infinity by a curve in $I(f)$. However, in [41], an example of an entire function was given for which this is not true, even though, by the results in [30], Eremenko's conjecture is satisfied for this function. This disproves a stronger form of Eremenko's conjecture. Among further results related to the escaping set of an entire function we mention [23, 26, 32, 42].

For a meromorphic function f the set $I(f)$ was first considered by Domínguez [10]. She proved that again $I(f) \neq \emptyset$, $\partial I(f) = J(f)$ and $I(f) \cap J(f) \neq \emptyset$. On the other hand, in this situation $\overline{I(f)}$ need not have unbounded components. For example, for $f(z) = \frac{1}{2} \tan z$ we have $\overline{I(f)} = J(f)$, and this set is totally disconnected.

In general it can be said that the set $I(f)$ is much less useful for meromorphic functions with infinitely many poles than for entire functions. For example, in many of the recent studies (e.g., [35, 36]) of the set $I(f)$ for entire f it turned out to be useful to consider the rate of escape

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to infinity for points in $I(f)$. For functions with infinitely many poles such escape rates are of limited use, since a point can escape to infinity arbitrarily fast by "jumping from pole to pole". However, some of the results about entire functions have been carried over to meromorphic functions with finitely many poles [10, 35, 38, 39].

Here we single out a large class of meromorphic functions which may have infinitely many poles, but which still have dynamical properties similar to entire functions. More specifically, we consider meromorphic functions which have a direct singularity over infinity. (The definition of a direct singularity will be given at the beginning of Section 2.)

Theorem 1.1. *Let f be a meromorphic function with a direct singularity over infinity. Then $I(f)$ has an unbounded component.*

Theorem 1.2. *Let f be a meromorphic function with a direct singularity over infinity. Then $I(f) \cap J(f)$ contains continua.*

In general, $I(f) \cap J(f)$ need not have unbounded components, but the case where this does not happen can be characterized. Recall that if f is a transcendental meromorphic function and if U is a component of $F(f)$, then there exists, for each $n \in \mathbb{N}$, a component of $F(f)$, which we call U_n, such that $f^n(U) \subset U_n$. If $U_m \neq U_n$ whenever $m \neq n$, then U is called a *wandering domain*. We use the name *Baker wandering domain* to denote a wandering domain U of $F(f)$ such that, for n large enough, U_n is a bounded multiply connected component of $F(f)$ which surrounds 0, and $U_n \to \infty$ as $n \to \infty$. The first example of such a wandering domain was given by Baker [1].

Theorem 1.3. *Let f be a meromorphic function with a direct singularity over infinity. Then $I(f) \cap J(f)$ has an unbounded component if and only if f has no Baker wandering domain.*

The proofs of the above results for entire functions (and also the proof that $I(f) \neq \emptyset$ for entire f) use Wiman-Valiron theory or the maximum principle and thus do not carry over to meromorphic functions with poles. Our main tools are results whose statements are similar to those of Wiman-Valiron theory [18], or of Macintyre's theory of flat regions [25]. We will develop these results in Section 2, but the methods are quite different from those of Wiman, Valiron and Macintyre.

An important special class of meromorphic functions is the Eremenko-Lyubich class B consisting of those functions for which there exists $R > 0$ such that the inverse function of f has no singularity over $\{z \in \mathbb{C} : |z| > R\}$. The main tool used to study this class of functions is the logarithmic change of variable, introduced to the subject by Eremenko and Lyubich [14]. In particular, they proved that if a transcendental entire function f belongs to the class B, then $I(f) \subset J(f)$.

Entire functions in the class B have a logarithmic singularity over infinity. Many results proved about the class B hold more generally for functions with a logarithmic singularity over infinity, with proofs carrying over to this more general setting without change. This observation is not new. For example, the results in [32, 41] are already stated for functions with a logarithmic singularity over infinity, and not just for entire functions in class B.

However, there are some results for class B where the generalization from "entire in class B" to "meromorphic with a logarithmic singularity" is not so obvious. In particular, this applies to the following result, where the proof for entire functions used Wiman-Valiron theory.

Theorem 1.4. *Let f be a meromorphic function with a logarithmic singularity over infinity. Then the upper box dimension of $I(f) \cap J(f)$ is 2 and the packing dimension of $J(f)$ is 2.*

Theorem 1.4 generalizes [37, Theorem 1.1] and we briefly outline here the changes in the proof needed in the present context. Similarly we sketch the changes needed in the proof of [39, Theorem 3] to give the following result.

Theorem 1.5. *Let f be a meromorphic function with a logarithmic singularity over infinity. Then the Hausdorff dimension of $J(f)$ is strictly greater than 1.*

The paper is organized as follows. In Section 2 we recall the classification of singularities of the inverse function and state the results of Wiman-Valiron type which will be used when dealing with direct singularities over infinity. The proof of these results will be deferred until Sections 9–12. In Section 3 we discuss iteration within a "tract" associated to the direct singularity over infinity and prove two results about fast escaping points whose orbits lie eventually in the tract (Theorems 3.2 and 3.3) from which Theorem 1.1 immediately follows. While the results of Section 3 only require that the function is defined in a "tract", the following two sections are devoted to functions meromorphic in the plane. In particular, Theorems 1.2 and 1.3 will be corollaries to two results (Theorems 5.1 and 5.3) proved in Section 5. In Section 6 we consider functions with a logarithmic tract and prove results from which Theorems 1.4 and 1.5 follow. In Section 7 we discuss results of Teichmüller and Selberg which show that, under suitable additional hypotheses, a meromorphic function has a logarithmic or direct singularity if it has few poles in the sense of Nevanlinna theory. Some examples will be discussed in Section 8.

Finally we note that the results of Section 2 may be applied not only in complex dynamics, but also to complex differential equations. One such application will be given in Section 13.

Remark. After completion of the paper, Alex Eremenko kindly informed us that the idea of using Wiman-Valiron theory for meromorphic functions with direct tracts appears already in his paper [11] dealing with differential equations of Briot-Bouquet type. This paper contains a statement equivalent to our Theorem 2.2, but according to A. Eremenko his proof of this statement contains a gap. Our Theorem 2.2 fills this gap.

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2. Direct and logarithmic tracts

We denote the open disc of radius r around a point $a \in \mathbb{C}$ by $D(a,r)$ and the closed disc by $\overline{D}(a,r)$. The open disc of radius r around a point $a \in \widehat{\mathbb{C}} = \mathbb{C} \cup \{\infty\}$ with respect to the spherical metric is denoted by $D_\chi(a,r)$.

We recall the classification of singularities of the inverse function of a meromorphic function due to Iversen [21]; see [28, p. 289]. Let f be meromorphic in the plane and let $a \in \widehat{\mathbb{C}}$. For $r > 0$ let U_r be a component of the preimage $f^{-1}(D_\chi(a,r))$, chosen in such a way that $r_1 < r_2$ implies $U_{r_1} \subset U_{r_2}$. Then there are two possibilities:

(a) $\bigcap_{r>0} U_r = \{z\}$ for some $z \in \mathbb{C}$, or
(b) $\bigcap_{r>0} U_r = \emptyset$.

In case (a) we have $a = f(z)$. If $a \in \mathbb{C}$ and $f'(z) \neq 0$ or if $a = \infty$ and z is a simple pole of f, then z is called an *ordinary point.* If $a \in \mathbb{C}$ and $f'(z) = 0$ or if $a = \infty$ and z is a multiple pole of f, then z is called a *critical point* and a is called a *critical value.* We also say f^{-1} has an *algebraic singularity* over a.

In case (b) we say that our choice $r \mapsto U_r$ defines a *transcendental singularity* of f^{-1} over a. A transcendental singularity over a is called *direct* if for some $r > 0$ we have $f(z) \neq a$ for $z \in U_r$. Otherwise it is called *indirect.* A direct singularity is called *logarithmic* if the restriction $f : U_r \to D_\chi(a,r)\backslash\{a\}$ is a universal covering for some $r > 0$.

Note that in case (b) there exists a curve γ tending to ∞ such that $f(z)$ tends to a as z tends to infinity along γ. A value a for which such a curve exists is called an *asymptotic value.* In turn, if a is an asymptotic value and γ is the corresponding curve, then by choosing U_r as the component of $f^{-1}(D_\chi(a,r))$ which contains the "tail" of this curve we obtain a transcendental singularity over a. Thus we see that f has a transcendental singularity over a if and only if a is an asymptotic value.

We note that over a point a there can be several singularities, of the same or of different types, and there may also be ordinary points over a. For example, the function $f(z) = \Gamma(z)\exp(z^2)$ has two transcendental singularities over ∞, with corresponding asymptotic paths being the positive and negative real axes. The singularity corresponding to the positive real axis is direct while the one corresponding to the negative real axis is indirect.

The domains U_r appearing in the definition of a direct or logarithmic singularity are called *tracts*. In some of our results we will work only with the restriction of f to a tract, and it will be irrelevant how f behaves outside the tract. This motivates the following definition.

Definition 2.1. Let D be an unbounded domain in $\mathbb{C}$ whose boundary consists of piecewise smooth curves. Suppose that the complement of D is unbounded. Let f be a complex-valued function whose domain of definition contains the closure $\overline{D}$ of D. Then D is called a *direct tract* of f if f is holomorphic in D and continuous in $\overline{D}$ and if there exists $R > 0$ such that $|f(z)| = R$ for $z \in \partial D$ while $|f(z)| > R$ for $z \in D$. If, in addition, the restriction $f : D \to \{z \in \mathbb{C} : |z| > R\}$ is a universal covering, then D is called a *logarithmic tract* of f.

Note that if f is meromorphic in the plane with a direct singularity over ∞, then f has a direct tract. In particular, every transcendental entire function has a direct tract. Similarly, if f has a logarithmic singularity over ∞, then f has a logarithmic tract.

We remark that the Denjoy-Carleman-Ahlfors Theorem [28, Section XI.4] says that a meromorphic function of finite order ρ can have at most $\max\{1, 2\rho\}$ direct singularities, and thus at most $\max\{1, 2\rho\}$ direct tracts, for each fixed value of R. In particular, a meromorphic function of order less than 1 can have at most one direct tract. Also, this holds if f is meromorphic and there is a sequence of Jordan curves γ_n surrounding the origin and tending to ∞ on which f tends to ∞. In particular, this is the case for a meromorphic function with a Baker wandering domain.

For a nonconstant subharmonic function $v : \mathbb{C} \to [0, \infty)$ the function

$$B(r, v) = \max_{|z|=r} v(z) \tag{2.1}$$

is increasing, convex in $\log r$ and tends to ∞ as r tends to ∞; see [20, Chapter 2]. Hence

$$a(r, v) = \frac{dB(r, v)}{d \log r} = rB'(r, v) \tag{2.2}$$

exists except perhaps for a countable set of r-values, and $a(r, v)$ is nondecreasing.

Note that if f, D, R are as in the above definition, then the function $v : \mathbb{C} \to [0, \infty)$ defined by

$$v(z) = \begin{cases} \log \dfrac{|f(z)|}{R} & \text{if } z \in D, \\ 0 & \text{if } z \notin D, \end{cases} \tag{2.3}$$

is subharmonic. While for a nonconstant function v subharmonic in the plane we only know [19, Theorem 2.14] that

$$\lim_{r\to\infty} \frac{B(r, v)}{\log r} > 0,$$

functions of the form (2.3) have faster growth.

Theorem 2.1. *Let D be a direct tract of f and let v be defined by* (2.3)*. Then*

$$\lim_{r\to\infty} \frac{B(r, v)}{\log r} = \infty \tag{2.4}$$

and

$$\lim_{r\to\infty} a(r, v) = \infty. \tag{2.5}$$

This result is due to Fuchs [16]. He proved only (2.4), but (2.5) follows from (2.4) since

$$a(r, v) \ge \frac{B(r, v) - B(r_0, v)}{\log(r/r_0)} \tag{2.6}$$

for $r > r_0 > 0$ by (2.2). We include a proof of Theorem 2.1 in Section 9 for completeness and because we use the techniques in the proof later in the paper.

We mention that $a(r, v)$ cannot only be estimated in terms of $B(r, v)$ from below as in (2.6), but also from above; see Lemma 6.10.

Our main tool to study functions with a direct tract is the following result, which is the main result in the paper. Here we say that a set $F \subset [1, \infty)$ has *finite logarithmic measure* if $\int_F dt/t < \infty$. We also put

$$M_D(r) = \max_{|z|=r, z \in D} |f(z)| = \exp B(r, v). \tag{2.7}$$

Theorem 2.2. *Let D be a direct tract of f and let $\tau > \frac{1}{2}$. Let v be defined by* (2.3) *and let z_r be a point satisfying $|z_r| = r$ and $v(z_r) = B(r, v)$. Then there exists a set $F \subset [1, \infty)$ of finite logarithmic measure such that if $r \in [1, \infty) \setminus F$, then $D(z_r, r/a(r, v)^\tau) \subset D$. Moreover,*

$$f(z) \sim \left(\frac{z}{z_r}\right)^{a(r,v)} f(z_r) \quad \textit{for} \quad z \in D\left(z_r, \frac{r}{a(r, v)^\tau}\right) \tag{2.8}$$

and

$$|f(z)| \sim M_D(|z|) \quad \textit{for} \quad z \in D\left(z_r, \frac{r}{a(r, v)^\tau}\right) \tag{2.9}$$

as $r \to \infty$, $r \notin F$.

We note that it follows from (2.8) that if $k \in \mathbb{N}$, then

$$f^{(k)}(z) \sim \left(\frac{a(r, v)}{z}\right)^k \left(\frac{z}{z_r}\right)^{a(r,v)} f(z_r) \quad \text{for} \quad z \in D\left(z_r, \frac{r}{a(r, v)^\tau}\right) \tag{2.10}$$

as $r \to \infty$, $r \notin F$. We will need (2.10) in Section 6 and Section 13.

Since $a(r, v) \to \infty$ as $r \to \infty$ by Theorem 2.1, since $e^h - 1 \sim h$ as $h \to 0$ and since $\tau > \frac{1}{2}$ is arbitrary, we see that conclusion (2.8) in Theorem 2.2 can be replaced by

$$f(z_r e^h) \sim e^{a(r,v)h} f(z_r) \quad \text{for} \quad |h| \leq a(r, v)^{-\tau}, \tag{2.11}$$

again as $r \to \infty$, $r \notin F$.

The asymptotic relations (2.8), (2.10) and (2.11) are very similar to the main results of Wiman-Valiron theory (see, e.g., [18]), except that the central index is replaced by $a(r, v)$. However, our results do not even require f to be defined outside of the closure of the tract D whereas the Wiman-Valiron method requires that f is entire, since it is based on the Taylor series expansion. It follows from the example in [18, pp. 345–346] that we cannot take $\tau = \frac{1}{2}$ in Theorem 2.2.

It is not difficult to see that $z_r f'(z_r)/f(z_r)$ lies between the left and right derivative of $B(r, v)$ with respect to $\log r$ and thus

$$a(r, v) = \frac{z_r f'(z_r)}{f(z_r)}$$

if $a(r, v)$ exists. (Actually, by a result of Blumenthal (see [49, Section II.3]), the set of r-values where $a(r, v)$ does not exist is discrete if $v = \log |f|$ for some entire function f, and Blumenthal's result extends to the case where v is as in Theorem 2.2.) With the above expression for $a(r, v)$ the relation (2.11) was proved by Macintyre [25] for entire functions. While it is possible to relax the assumption that f is entire, it seems essential for his method that a certain disc around z_r is in the domain of definition of f. The key conclusion of Theorem 2.2 therefore is that $D(z_r, r/a(r, v)^\tau) \subset D$ and most of the proof of Theorem 2.2 is devoted to this fact.

As in Wiman-Valiron theory, an important consequence of (2.8) or (2.11) is the following result, which can be deduced from them for example by Rouché's theorem.

Theorem 2.3. *For each $\beta > 1$ there exists $\alpha > 0$ such that if f, D, v, z_r and F are as in Theorem 2.2 and if $r \notin F$ is sufficiently large, then*

$$\left\{ z \in \mathbb{C} : \frac{|f(z_r)|}{\beta} \leq |z| \leq \beta |f(z_r)| \right\} \subset f\left(D\left(z_r, \frac{\alpha r}{a(r,v)} \right) \right).$$

More precisely, $\log f$ *is univalent in* $D(z_r, \alpha r/a(r,v))$ *and for* $\gamma > \pi$ *the constant* α *can be chosen such that if* $r \notin F$ *is sufficiently large, then* $\log f\left(D(z_r, \alpha r/a(r,v))\right)$ *contains the rectangle*

$$\{z \in \mathbb{C} : |\operatorname{Re} z - \log |f(z_r)|| \leq \log \beta, |\operatorname{Im} z - \arg f(z_r)| \leq \gamma\},$$

where the branches of $\log$ *and* $\arg$ *are chosen such that* $\operatorname{Im}(\log f(z_r)) = \arg f(z_r)$.

Theorem 2.3 is sufficient for our purposes, but we note that this statement can be strengthened by working with the disc $D(z_r, r/a(r,v)^\tau)$, for $\frac{1}{2} < \tau < 1$.

3. Iteration in a tract

Wiman-Valiron theory was the main tool in Eremenko's proof [12] that $I(f) \neq \emptyset$ if f is a transcendental entire function. Using Theorem 2.3 instead of the Wiman-Valiron method in his argument, we obtain the following result.

Theorem 3.1. *Let D be a direct tract of f. Then there exists $z_0 \in D$ such that $f^n(z_0) \in D$ for all $n \in \mathbb{N}$ and $f^n(z_0) \to \infty$ as $n \to \infty$.*

We emphasize that we are not assuming here that f is defined in the whole plane, but only that the domain of definition of f contains $\overline{D}$. However, the result appears to be new even for entire f. Neither Eremenko's argument based on the Wiman-Valiron method nor Domínguez's argument [10] based on the maximum modulus and a theorem of Bohr seem to give, for entire f, the existence of a point $z_0 \in I(f)$ such that $f^n(z_0)$ is in the same direct tract for all n. The conclusion of Theorem 3.1 is known, however, if D is a logarithmic tract [31].

Proof of Theorem 3.1. We adapt the argument of Eremenko. Let F be the exceptional set arising in Theorems 2.2 and 2.3, with $\beta = 4$ in Theorem 2.3, and α chosen accordingly. For large r we have $\left[\frac{1}{2}|f(z_r)|, 2|f(z_r)|\right] \setminus F \neq \emptyset$ and $|f(z_r)| > 4r$, by (2.4). Choosing $r_0 \notin F$ large we can thus inductively define an increasing sequence (r_n) satisfying

$$r_{n+1} \in \left[\tfrac{1}{2}|f(z_{r_n})|, 2|f(z_{r_n})|\right] \setminus F$$

and $r_n \to \infty$. We put $\overline{D}_n = \overline{D}\left(z_{r_n}, \alpha r_n/a(r_n, v)\right)$. Choosing r_0 large we have $\overline{D}_{n+1} \subset \left\{z \in \mathbb{C} : \frac{1}{4}|f(z_{r_n})| \leq |z| \leq 4|f(z_{r_n})|\right\}$ and thus, by Theorem 2.3, $\overline{D}_{n+1} \subset f\left(\overline{D}_n\right)$ for all $n \geq 0$. Inductively we see that there exists a closed set $C_n \subset \overline{D}_0$ such that $f^j(C_n) \subset \overline{D}_j$ for $0 \leq j \leq n$ and $C_{n+1} \subset C_n$ for all $n \geq 0$. Choosing $z_0 \in \bigcap_{n=1}^\infty C_n$ we see that $f^n(z_0) \in D$ for all $n \in \mathbb{N}$ and $f^n(z_0) \to \infty$ as $n \to \infty$. □

Using (2.9) we see that the chosen point z_0 actually satisfies

$$|f^{n+1}(z_0)| \sim M_D(|f^n(z_0)|) \tag{3.1}$$

as $n \to \infty$.

It follows from (2.4) that $M_D(\rho) > \rho$ for large ρ, say $\rho > \rho_0 > R$. Hence $M_D^n(\rho) \to \infty$ as $n \to \infty$ for $\rho > \rho_0$. For such ρ we define

$$A(f, D, \rho) = \{z \in D : f^n(z) \in D \text{ and } |f^n(z)| \geq M_D^n(\rho) \text{ for all } n \in \mathbb{N}\}.$$

In contrast to similar definitions in [6, 36], the set $A(f, D, \rho)$ depends on ρ. Note that $f^n(z) \to \infty$ as $n \to \infty$ for $z \in A(f, D, \rho)$. In particular, if f is meromorphic in $\mathbb{C}$, then $A(f, D, \rho) \subset I(f)$.

The following result, proved below, strengthens Theorem 3.1.

Theorem 3.2. *Let D be a direct tract of f. Then $A(f, D, \rho) \neq \emptyset$.*

While the results in [6, 36] yield that for an entire function f there exists $z_0 \in \mathbb{C}$ satisfying $|f^n(z_0)| \geq M^n(\rho, f)$, where $M(r, f) = \max_{|z|=r} |f(z)|$, with the exponent n indicating iteration with respect to the first variable, the result that z_0 can be chosen such that $f^n(z_0)$ is in the same direct tract of f for every $n \in \mathbb{N}$ is again new even for entire f.

Theorem 3.3. *Let D be a direct tract of f. Then all components of $A(f, D, \rho)$ are unbounded.*

Proof of Theorem 3.2. We follow the proof of Lemma 2 in [6]. Fix $\rho > \rho_0$. Since $\log M_D(r) = B(r, v)$ is convex in $\log r$ we deduce from (2.4) that

$$M_D(2r) \geq 4M_D(r) \tag{3.2}$$

for large r, say $r \geq r_0 > \rho$. By Eremenko's argument there exists $z_0 \in D$ satisfying (3.1) such that $f^n(z_0) \in D$ for all $n \in \mathbb{N}$. We may assume that

$$|f^{n+1}(z_0)| \geq \tfrac{1}{2} M_D\left(|f^n(z_0)|\right) \tag{3.3}$$

for all $n \in \mathbb{N}$ and that

$$|z_0| \geq 2r_0, \tag{3.4}$$

because otherwise we can replace z_0 by $f^k(z_0)$ for a sufficiently large k. We shall prove by induction that

$$|f^n(z_0)| \geq 2M_D^n(\rho) \tag{3.5}$$

for all $n \geq 0$. Because of (3.4) and since $r_0 \geq \rho$ we see that (3.5) holds for $n = 0$. Suppose it holds for some $n \geq 0$. Combining this with (3.2) and (3.3) we deduce that

$$|f^{n+1}(z_0)| \geq \tfrac{1}{2} M_D\left(|f^n(z_0)|\right) \geq 2M_D\left(\tfrac{1}{2}|f^n(z_0)|\right) \geq 2M_D\left(M_D^n(\rho)\right) = 2M_D^{n+1}(\rho),$$

and thus (3.5) also holds with n replaced by $n + 1$. Of course, it follows from (3.5) that $z_0 \in A(f, D, \rho)$. □

Proof of Theorem 3.3. Let $z_0 \in A(f, D, \rho)$. We follow the argument in [36] and denote by L_n the component of $f^{-n}(\mathbb{C} \setminus D(0, M_D^n(\rho)))$ that contains z_0. Then L_n is closed and unbounded. In fact, since $|f(z)| = R < \rho \leq M_D^k(\rho)$ for $z \in \partial D$ and $0 \leq k < n$, we have $f^k(z) \in D$ for $z \in L_n$ and $0 \leq k < n$; in particular, $L_n \subset D$. We claim that $L_{n+1} \subset L_n$. Otherwise there exists $z' \in L_{n+1}$ with $f^n(z') \in D(0, M_D^n(\rho))$. This implies that $|f^{n+1}(z')| < M_D(M_D^n(\rho))) = M_D^{n+1}(\rho)$ so that $f^{n+1}(z') \notin \mathbb{C} \setminus D(0, M_D^{n+1}(\rho))$, contradicting $z' \in L_{n+1}$. Hence $L_{n+1} \subset L_n$. As in [36] we conclude that

$$K = \bigcap_{n=1}^{\infty} L_n \cup \{\infty\}$$

is a closed connected subset of $\widehat{\mathbb{C}}$ containing z_0 and ∞. It follows from the construction that we also have $K \setminus \{\infty\} \subset A(f, D, \rho)$. The component of $A(f, D, \rho)$ containing z_0 thus also contains a component of $K \setminus \{\infty\}$, and hence it is unbounded; see [29, p. 84]. □

Remark. The arguments of this section can be used to show that if a function f has N tracts $D_1, \ldots, D_N$ and if $(s_k)_{k \geq 0}$ is a sequence in $\{1, \ldots, N\}$, then there exists $z \in D_{s_0}$ such that $f^k(z) \in D_{s_k}$ for all $k \in \mathbb{N}$ and $f^k(z) \to \infty$ as $k \to \infty$. Moreover, the set of all z with this property has an unbounded component.

If f has infinitely many tracts $D_1, D_2, \ldots$, then not every sequence (s_k) in $\mathbb{N}$ is admissible. This occurs, for example, with the function $f(z) = \exp(\exp z)$. Noting that the tracts of $\exp(\exp z)$ are contained in half-strips $\{z \in \mathbb{C} : \operatorname{Re} z > 0, (2j - 1)\pi < \operatorname{Im} z < (2j + 1)\pi\}$ with $j \in \mathbb{Z}$, we see that the question of when a sequence (s_k) is admissible is closely connected to the concept of an allowable itinerary considered in [8, Section 3] and [9] and many other papers on exponential dynamics.

4. Meromorphic functions with a direct tract

Let f be a transcendental function meromorphic in the plane which has a direct tract D, and let $\rho > \rho_0$ be as in the previous section. We define

$$\begin{aligned} A(f,D) &= \{z \in \mathbb{C} : \text{ there exists } L \in \mathbb{N} \text{ such that } f^L(z) \in A(f,D,\rho)\} \\ &= \{z \in \mathbb{C} : \text{ there exists } L \in \mathbb{N} \text{ such that } f^{n+L}(z) \in D \\ &\qquad \text{and } |f^{n+L}(z)| \geq M_D^n(\rho) \text{ for all } n \in \mathbb{N}\}. \end{aligned}$$

The set $A(f,D)$ is a variation of the set of fast escaping points of an entire function:

$$A(f) = \{z \in \mathbb{C} : \text{ there exists } L \in \mathbb{N} \text{ such that } f^{n+L}(z) \geq M(R,f^n) \text{ for all } n \in \mathbb{N}\},$$

where R is so large that $J(f) \cap D(0,R) \neq \emptyset$. It is known [6, 36] that $A(f) \neq \emptyset$, $\partial A(f) = J(f)$, $A(f) \cap J(f) \neq \emptyset$ and $A(f)$ is completely invariant; moreover all components of $A(f)$ are unbounded. It can be shown that, in the definition of $A(f)$, we can replace $M(R,f^n)$ by $M^n(R,f)$. Note that for a given tract D, the set $A(f,D)$ may not be a subset of $A(f)$; see Remark 2 at the end of this section.

Using arguments as in [6, 36] one can show that the set $A(f,D)$ does not depend on the choice of ρ, as long as $\rho > \rho_0$. This justifies the notation where ρ is suppressed. It follows from Theorem 3.3 and arguments given in [36] that if f is entire, then all components of $A(f,D)$ are unbounded.

In this section and the next we prove a number of properties of $A(f,D)$ which are analogous to known properties of $A(f)$, when f is entire, and also to similar properties of meromorphic functions with a finite number of poles; see [6, 12, 36, 38]. Many proofs in these two sections are modifications of proofs in [38] and we do not always give full details.

We also define

$$Z(f) = \left\{z \in I(f) : \tfrac{1}{n}\log\log|f^n(z)| \to \infty \text{ as } n \to \infty\right\}$$

and we recall from [5, Lemma 7] that any periodic Fatou component of f does not meet $Z(f)$.

Theorem 4.1. *Let f be a transcendental meromorphic function with a direct tract D. Then the following properties hold:*

(a) *$A(f,D) \neq \emptyset$ and, for each $z \in A(f,D)$, there exists $L \in \mathbb{N}$ such that $f^L(z)$ lies in an unbounded closed connected subset of $A(f,D)$;*
(b) *$A(f,D)$ is completely invariant under f;*
(c) *$A(f,D) \subset Z(f)$.*

Proof. Part (a) follows from Theorems 3.2 and 3.3, and the definition of $A(f,D)$.

Part (b) follows easily from the fact that $M_D(r) > r$ for $r \geq \rho$.

Part (c) follows from part (b) and (2.4) which says that $\log M_D(r)/\log r \to \infty$ as $r \to \infty$. □

Our next result relates $A(f,D)$ to the Fatou set and Julia set of f.

Theorem 4.2. *Let f be a transcendental meromorphic function with a direct tract D. Then the following properties hold:*

(a) *if U is a component of $F(f)$ such that $U \cap A(f,D) \neq \emptyset$, then $U \subset A(f,D)$ and U is a wandering domain;*
(b) *$J(f) = \partial A(f,D)$;*
(c) *if f has no wandering domains, then $J(f) = \overline{A(f,D)}$.*

Proof. To prove part (a), let U be a component of $F(f)$ which meets $A(f,D)$. By the complete invariance of $A(f,D)$ we can assume that U meets $A(f,D,\rho)$, for some $\rho > 0$, at a point z_0 say. Let Δ_0 be a closed disc in U with centre z_0. Then

$$|f^n(z_0)| > M_D^n(\rho) \quad \text{and} \quad f^n(z_0) \in D, \quad \text{for } n = 0, 1, \ldots.$$

By [35, Theorem 3(a)], for example, there exists $C > 0$ such that for all $z \in \Delta_0$, and $n = 0, 1, \ldots$, we have

$$|f^n(z)| \geq |f^n(z_0)|^{1/C} > M_D^n(\rho)^{1/C}.$$

Since $\log M_D^n(\rho)/\log M_D^{n-1}(\rho) \to \infty$ as $n \to \infty$, we deduce that, for all $z \in \Delta_0$ and all sufficiently large n,

$$|f^n(z)| > M_D^{n-1}(\rho) > R \quad \text{so} \quad f^n(z) \in D,$$

since $|f| = R$ on ∂D. Hence, for some $m \in \mathbb{N}$ we have $f^m(z) \in A(f, D, \rho)$. Thus $z \in A(f, D)$, so $\Delta_0 \subset A(f, D)$. It follows by a compactness argument that $U \subset A(f, D)$, as required. Hence, by Theorem 4.1(c), U is a wandering domain.

The proof of part (b) is identical to the proof of [38, Theorem 2(c)] and part (c) follows immediately from parts (a) and (b). □

Remark 1. One consequence of Theorems 4.1 and 4.2 is that if f is a transcendental meromorphic function with a direct tract D, then none of the sets $J(f)$, $A(f, D)$, $Z(f)$ and $I(f)$ can contain a free Jordan arc; see [38, Theorem 6] for details.

Remark 2. Taking $R > e+1$ we see that $f(z) = \exp(-z)+\exp(\exp(z))$ has one direct tract D in the left half-plane and infinitely many direct tracts in the right half-plane. It is not difficult to see that $A(f, D) \cap A(f) = \emptyset$. On the other hand, denoting by D' the direct tract that contains all large positive real numbers, we see that $A(f, D') \subset A(f)$.

There are also examples of entire functions f for which $A(f, D) \cap A(f) = \emptyset$ for every direct tract D. More specifically, there exists an entire function f with exactly two tracts D_1 and D_2 such that $A(f, D_j) \cap A(f) = \emptyset$ for $j = 1, 2$. We only indicate very briefly how such a function can be constructed. Let $\theta_j : (1, \infty) \to (0, \frac{1}{2}\pi)$ be continuous for $j = 1, 2$ and let $\Omega_1 = \{re^{i\varphi} : r > 1, |\varphi| < \theta_1(r)\}$ and $\Omega_2 = \{re^{i\varphi} : r > 1, |\varphi - \pi| < \theta_2(r)\}$. There are several techniques for constructing an entire function f which has two tracts D_1 and D_2 which are "close" to Ω_1 and Ω_2, provided the functions θ_1 and θ_2 are sufficiently "nice". One such technique is the Kjellberg-Kennedy-Katifi approximation method described in [19, Section 10.5], another one is via Cauchy integrals (see [15, Section VI.4] or [46]). One also has control over the growth of f in the tracts by choosing the sizes of $\theta_j(r)$ as r varies. In particular, one can arrange that $M_{D_1}(r)$ is much bigger than $M_{D_2}(r)$ in certain intervals and much smaller than $M_{D_2}(r)$ in other intervals. In this way it is possible to construct f such that $M_{D_1}^n(\rho)$ and $M_{D_2}^n(\rho)$ both grow much more slowly than $M^n(\rho, f)$ as $n \to \infty$. This then implies that $A(f, D_j) \cap A(f) = \emptyset$ for $j = 1, 2$.

5. Meromorphic functions and Baker wandering domains

In this section we prove some results about Baker wandering domains which, in particular, contain Theorems 1.2 and 1.3 as special cases. Recall that if U is a component of $F(f)$, then U_n denotes the component of $F(f)$ such that $f^n(U) \subset U_n$, and that a Baker wandering domain is a component U of $F(f)$ such that, for n large enough, U_n is a bounded multiply connected component of $F(f)$ which surrounds 0, and $U_n \to \infty$ as $n \to \infty$. We use the notation $\widetilde{U}$ to denote the union of a set U and its bounded complementary components. If $\widetilde{U} = U$, then we say that U is *full*.

Theorem 5.1. *Let f be a transcendental meromorphic function with a direct tract D. If f has a Baker wandering domain U, then*

(a) *$\overline{U} \subset A(f, D)$, more precisely, there exist $N \in \mathbb{N}$ and $\rho > 0$ such that, for $n \geq N$,*

$$U_n \subset \widetilde{U_{n+1}} \quad \text{and} \quad \overline{U_n} \subset A(f, D, \rho);$$

(b) *$\overline{U_n} \subset D$, for $n \geq N$, so D is the only direct tract of f and it has no unbounded complementary components;*

(c) *$A(f, D) \cap J(f)$ contains infinitely many bounded continua;*

(d) *$A(f,D)$ has exactly one unbounded component, as do $Z(f)$ and $I(f)$.*

Thus if f has a direct tract D and a Baker wandering domain, then $A(f,D)$ has a single unbounded component which must meet both $F(f)$ and $J(f)$. We remark that there exist entire functions for which $A(f,D)$ and $I(f)$ are connected, and $I(f) \subset J(f)$; see [40].

Next we give a sufficient condition for Baker wandering domains to exist.

Theorem 5.2. *Let f be a transcendental meromorphic function with a direct tract D. Then there is a constant $r_0 > 0$ such that if U is a component of $F(f)$ which contains a Jordan curve surrounding $\{z \in \mathbb{C} : |z| = r_0\}$, then U is a Baker wandering domain.*

In Theorem 5.1, we showed that $A(f,D) \cap J(f)$ contains continua if f has a Baker wandering domain. We now show that this conclusion also holds if f does not have Baker wandering domains.

Theorem 5.3. *Let f be a transcendental meromorphic function with a direct tract D and suppose that f has no Baker wandering domains. Then $A(f,D) \cap J(f)$ has at least one unbounded component, as do $Z(f) \cap J(f)$ and $I(f) \cap J(f)$.*

Remark. In particular, it follows from Theorem 5.3 that $J(f)$ has an unbounded component, and it is this statement which will be proved first.

To prove these results we use the concept of an outer sequence. Starting from a full open set G_0 (not necessarily connected), we carry out the following process. First, form $B_0 = G_0 \cap D$, then take the image under f and finally fill in any bounded complementary components to obtain the full open set $G_1 = \widetilde{f(B_0)}$. Then we repeat this process to obtain a sequence (G_n) of full open sets, so long as this is possible. The following lemma shows that if G_0 is a large enough Jordan domain, then (G_n) is a sequence of Jordan domains whose boundaries tend to ∞. If G_0 satisfies the hypotheses of Lemma 5.1, then we say that the sets $E_n = \mathbb{C} \setminus G_n$ form an *outer sequence* for (f,D).

Lemma 5.1. *Let f be a transcendental meromorphic function with a direct tract D and let $M_D(r)$ be defined by (2.7). Then there is a constant $r_0 > 0$ such that $D \cap \{z \in \mathbb{C} : |z| = r_0\} \neq \emptyset$ and if G_0 is a Jordan domain with boundary γ_0 that surrounds $\{z \in \mathbb{C} : |z| = r_0\}$, then the corresponding sets G_n are Jordan domains with the following properties.*

(a) *For $n = 0, 1, \dots$, the Jordan curve $\gamma_n = \partial G_n$ surrounds $\{z \in \mathbb{C} : |z| = r_n\}$, where*

$$r_{n+1} > 4M_D(r_n/4) > 2r_n, \quad \text{for } n = 0, 1, \dots.$$

(b) *For $n = 0, 1, \dots$,*

$$\gamma_{n+1} \subset f(\gamma_n \cap D).$$

(c) *For $n = 0, 1, \dots$, any component of $f^{-1}(E_{n+1})$ which meets $E_n \cap D$ lies in $E_n \cap D$.*

Proof. Let R, v, z_r and F be as in Theorem 2.3, with $\beta = 4$ and the corresponding value of α. Then choose $r_0 > 0$ such that $D \cap \{z \in \mathbb{C} : |z| = r_0/4\} \neq \emptyset$,

$$\text{each interval of the form } (r, 2r), r \geq r_0/4, \text{ contains a point outside } F, \tag{5.1}$$

which is possible since F has finite logarithmic measure,

$$\frac{\alpha}{a(r,v)} < 1, \quad \text{for } r > r_0/4, \tag{5.2}$$

which is possible since $a(r,v) \to \infty$ as $r \to \infty$,

$$r_0 > R, \tag{5.3}$$

$$\text{the conclusion of Theorem 2.3 holds for } r \in (r_0/4, \infty) \setminus F, \tag{5.4}$$

$$M_D(r) > 2r, \quad \text{for } r \geq r_0/4, \tag{5.5}$$

which is possible since $B(r)/\log r = \log M_D(r)/\log r \to \infty$ as $r \to \infty$.

Now by (5.1) we can choose $\rho_0 \in (r_0/4, r_0/2) \setminus F$ and ζ_0 such that $\zeta_0 = z_{\rho_0}$. Then $|f(\zeta_0)| = M_D(\rho_0)$ and, by (5.4),

$$f(A_0) \supset \left\{z \in \mathbb{C} : \tfrac{1}{4}M_D(\rho_0) < |z| < 4M_D(\rho_0)\right\},$$

where $A_0 = D(\zeta_0, \alpha\rho_0/a(\rho_0, v))$. Recall from Theorem 2.2 that $A_0 \subset D$. Also, $A_0 \subset G_0$, since $\alpha\rho_0/a(\rho_0, v) < \rho_0 < r_0/2$, by (5.2). Thus $A_0 \subset B_0 = G_0 \cap D$, so

$$f(B_0) \supset \left\{z \in \mathbb{C} : \tfrac{1}{4}M_D(\rho_0) < |z| < 4M_D(\rho_0)\right\}.$$

Clearly $f(B_0)$ is bounded and there are no components of $f(B_0)$ which lie entirely outside the above annulus, since any such component would have to meet the circle $\{z \in \mathbb{C} : |z| = R\}$. Thus $G_1 = \widetilde{f(B_0)}$ is a Jordan region and $\gamma_1 = \partial G_1$ is a Jordan curve which lies outside $\{z \in \mathbb{C} : |z| = r_1\}$, where $r_1 = 4M_D(\rho_0)$. Also, by (5.3) and (5.5), we have

$$r_1 = 4M_D(\rho_0) > 4M_D(r_0/4) > 2r_0 > R.$$

Thus $\gamma_1 \subset f(\gamma_0 \cap D)$ because $\partial f(B_0) \subset f(\partial B_0)$ and $|f| = R$ on $\partial B_0 \setminus \gamma_0$.

Recall that, for $n = 0, 1, \ldots$, the set E_n denotes the complement of G_n. If K is any component of $f^{-1}(E_1)$ which meets $E_0 \cap D$, then K must lie entirely in D (because $|f| = R < r_1$ on ∂D) and cannot meet B_0 (since $f(B_0) \cap E_1 = \emptyset$). Thus K is a subset of $E_0 \cap D$.

We can now carry out this process with γ_1 and r_1 in place of γ_0 and r_0, and continue repeatedly to produce the required sequences (G_n), (γ_n), (r_n) and (E_n) which satisfy parts (a)–(c). □

Remark. The sequences (G_n), (γ_n) and (E_n) defined in Lemma 5.1 are related to sequences of sets introduced in [36] and [38]. For a transcendental entire function f the set

$$B(f) = \{z \in \mathbb{C} : \text{ there exists } L \in \mathbb{N} \text{ such that } f^{n+L}(z) \notin \widetilde{f^n(\Delta)}, \text{ for } n \in \mathbb{N}\},$$

where Δ is an open disc that meets $J(f)$, was defined in [36] and proved equal to the set $A(f)$ defined earlier. The above sequence (G_n) is a modification of the sequence $\left(\widetilde{f^n(\Delta)}\right)$. The definition of $B(f)$ was extended to transcendental meromorphic functions with a finite number of poles in [38] using the concept of an outer sequence.

Outer sequences give us an alternative way to prove that points are in $A(f, D)$.

Lemma 5.2. *Suppose that f, D, E_n and r_n are as in Lemma 5.1. If z satisfies*

$$f^n(z) \in E_n \cap D, \quad \text{for } n = 0, 1, 2, \ldots,$$

then

$$|f^n(z)| \geq 4M_D^n(r_0/4), \quad \text{for } n = 0, 1, 2, \ldots,$$

so $z \in A(f, D, \rho)$, where $\rho = r_0/4$.

Proof. Since

$$|f^n(z)| \geq r_n, \quad \text{for } n = 0, 1, 2, \ldots,$$

and, by Lemma 5.1(a) and induction,

$$r_n \geq 4M_D^n(r_0/4), \quad \text{for } n = 0, 1, \ldots,$$

the result follows. □

Remark. We could use the outer sequences in Lemma 5.1 to define $B(f, D)$ in a similar way to the definition of $B(f)$ in [38], and then use Lemma 5.2 to prove that $B(f, D) = A(f, D)$.

Proof of Theorem 5.1. Let $U_0 = U$ be a Baker wandering domain. By the definition of a Baker wandering domain, we deduce from Theorem 3.3 and Theorem 4.2(a) that $U_n \subset A(f, D)$ for all sufficiently large n, so $U_0 \subset A(f, D)$ by Theorem 4.1(b). To prove that $\overline{U_0} \subset A(f, D)$, we argue more carefully as follows. By the complete invariance of $A(f, D)$, we can renumber U_n so that

$$U_n \text{ surrounds } \{z \in \mathbb{C} : |z| = r_0\}, \quad \text{for } n = 0, 1, \ldots,$$

where r_0 satisfies the hypotheses of Lemma 5.1. In particular, for $n = 0, 1, \ldots$, we have $U_n \cap D \neq \emptyset$, by Lemma 5.1, and $|f(z)| > r_0 > R$, for $z \in U_n$, so $\overline{U_n} \subset D$.

Then take a Jordan curve γ_0 in U_0 surrounding 0 and define γ_n, G_n and E_n as in Lemma 5.1 and $B_n = G_n \cap D$. Clearly $\gamma_n \subset U_n$, for $n = 0, 1, \ldots$. We now show that

$$U_n \subset \widetilde{U_{n+1}}, \quad \text{for } n \geq 0.$$

To prove this we show that, for $n \geq 0$, the curve γ_{n+1} lies outside γ_n. If this is false, then γ_{n+1} lies inside γ_n and hence $G_{n+1} \subset G_n$. But then

$$G_{n+2} = \widetilde{f(B_{n+1})} \subset \widetilde{f(B_n)} = G_{n+1} \subset G_n.$$

By induction, $G_m \subset G_n$, for $m > n$, which contradicts Lemma 5.1(a).

Thus $\overline{U_{n+1}} \subset E_n \cap D$, for $n = 0, 1, \ldots$, so $\overline{U_{n+1}} \subset A(f, D, r_0/4)$, for $n = 0, 1, \ldots$, by Lemma 5.2, as required.

The fact that $\overline{U_n} \subset D$, for $n = 0, 1, \ldots$, implies that D is the only tract of f and all its complementary components are bounded.

Part (c) holds because $\partial U_n \subset A(f, D) \cap J(f)$, for $n = 0, 1, \ldots$.

Part (d) follows from part (a), Theorem 3.3 and Theorem 4.1(c). □

Proof of Theorem 5.2. Let r_0 be the constant in Lemma 5.1. Denote the given Jordan curve by γ_0 and define the corresponding outer sequence (E_n) as in Lemma 5.1. Then each γ_n meets $A(f, D)$, by Theorem 3.3 and Theorem 4.1(b). Thus the Fatou components U_n are wandering by Theorem 4.2(a) and hence they are disjoint. Since each component U_n contains the curve γ_n and does not meet any γ_m for $m \neq n$, it follows from Lemma 5.1(a) that $U = U_0$ is a Baker wandering domain. □

The proof of Theorem 5.3 is longer. We begin by showing that $J(f)$ has at least one unbounded component. First recall from [38, Section 6] some general properties of the family $\mathcal{J}$ of components J of $J(f)$. For $J \in \mathcal{J}$, we again use the notation $\widetilde{J}$ to denote the union of J and its bounded complementary components, and we associate to each $J \in \mathcal{J}$ the set

$$\Omega_J = \bigcup \left\{ \widetilde{J'} : J' \in \mathcal{J}, \widetilde{J} \subset \widetilde{J'} \right\}.$$

Note that $\partial\Omega_J$ is a subset of $J(f)$ since if $z \in \partial\Omega_J$, then each open neighbourhood of z must meet some component of $J(f)$. Also, each Ω_J is connected. The following lemma shows that each set Ω_J is full.

Lemma 5.3. *Let f be meromorphic and for each $J \in \mathcal{J}$ let Ω_J be defined as above.*

(a) *If Ω_J is bounded, then Ω_J is compact and full.*
(b) *If Ω_J is unbounded, then*
  (i) *either $\Omega_J = \widetilde{J'}$ for some unbounded $J' \in \mathcal{J}$ so Ω_J is closed and full,*
  (ii) *or Ω_J is the union of a sequence of bounded sets of the form $\widetilde{J_n}$, $n = 0, 1, \ldots$, where $J_n \in \mathcal{J}$ and $\widetilde{J_1} \subset \widetilde{J_2} \subset \cdots$, and Ω_J is open and full.*

Lemma 5.3 was proved in [38, Lemma 6 and its proof] under the assumption that f has a finite number of poles, but it was remarked there that it is essentially a topological result about a closed set in $\mathbb{C}$.

Next we show that, with f and D as above, the sets Ω_J cannot all be bounded.

Lemma 5.4. *Let f be a transcendental meromorphic function with a direct tract. Then Ω_J is unbounded for at least one $J \in \mathcal{J}$.*

Proof. Suppose that all Ω_J, $J \in \mathcal{J}$, are bounded. Then all Ω_J, $J \in \mathcal{J}$, are compact and full by Lemma 5.3(a). Also note that any two Ω_J are either disjoint or identical. We consider the set

$$U = \mathbb{C} \setminus \bigcup \{\Omega_J : J \in \mathcal{J}\}.$$

Then it is a topological result (see [38, proof of Lemma 7]) that U is nonempty, open and connected.

Now U is evidently an unbounded subset of $F(f)$. Moreover U must be a component of $F(f)$, indeed the only unbounded component of $F(f)$ because any unbounded connected set must meet U. If D is a tract of f and v is the corresponding subharmonic function defined in (2.3), then v and hence f has asymptotic value ∞ by Iversen's theorem for subharmonic functions [20, Theorem 4.17]. Therefore U must be invariant under f, since $f(U)$ cannot be bounded. But we also know that $U \cap A(f,D) \neq \emptyset$, by Theorem 3.3, so we obtain a contradiction to Theorem 4.2(a). □

Next we relate Lemma 5.3 to the existence of Baker wandering domains.

Lemma 5.5. *Let f be a transcendental meromorphic function with a direct tract. Then f has a Baker wandering domain if and only if Lemma 5.3(b)(ii) holds with $\Omega_J = \mathbb{C}$, for some $J \in \mathcal{J}$.*

Proof. It is clear that if f has a Baker wandering domain, then $\mathbb{C}$ is the union of bounded sets of the form $\widetilde{J_n}$, $n = 0, 1, \ldots,$ where

$$J_n \in \mathcal{J} \quad \text{and} \quad \widetilde{J_1} \subset \widetilde{J_2} \subset \cdots . \tag{5.6}$$

Suppose, on the other hand, that, for some $J \in \mathcal{J}$, the set $\Omega_J = \mathbb{C}$ is the union of bounded sets of the form $\widetilde{J_n}$, $n = 1, 2, \ldots,$ such that (5.6) holds.

Choose N so large that $\{z \in \mathbb{C} : |z| = r_0\} \subset \widetilde{J_N}$, where r_0 satisfies the hypotheses of Lemma 5.1. Then $E = J(f) \cup \{\infty\}$ is a closed subset of $\hat{\mathbb{C}}$ having J_N and J_{N+1} amongst its components. Thus there is a simple polygon γ_0 separating J_N and J_{N+1}, and lying in $F(f)$. Since γ_0 surrounds $\{z \in \mathbb{C} : |z| = r_0\}$, we deduce from Theorem 5.2 that the component U of $F(f)$ which contains γ_0 is a Baker wandering domain, as required. □

It follows from Lemmas 5.3, 5.4 and 5.5 that if f and D are as above and f has no Baker wandering domains, then for some $J \in \mathcal{J}$ the set Ω_J is unbounded and either $\Omega_J = \widetilde{J'}$, for some $J' \in \mathcal{J}$, or Ω_J is a union of bounded sets of the form $\widetilde{J_n}$, $J_n \in \mathcal{J}$, and $\Omega_J \neq \mathbb{C}$. In the former case, $J(f)$ certainly has an unbounded component, namely J'. In the latter case, Ω_J is unbounded, open, connected and full. As in [38, page 240], we then deduce that the boundary of any complementary component of Ω_J is an unbounded subset of $J(f)$, as required. Thus in either case $J(f)$ has an unbounded component.

To complete the proof of Theorem 5.3, we need another topological lemma.

Lemma 5.6. *Let f be a transcendental meromorphic function with a direct tract D and suppose that $(E_n)_{n\geq 0}$ is an outer sequence for (f,D), given by Lemma 5.1, with corresponding Jordan curves γ_n. For $n \geq 1$, let K_n be a closed subset of $J(f) \cap E_n$ which meets γ_n and whose components are unbounded. Then $K_{n-1} = f^{-1}(K_n) \cap E_{n-1} \cap D$ is a closed subset of $J(f) \cap E_{n-1} \cap D$ which meets $\gamma_{n-1} \cap D$ and whose components are unbounded.*

Proof. First, $K_{n-1} = f^{-1}(K_n) \cap E_{n-1} \cap D$ is a closed subset of $J(f) \cap E_{n-1} \cap D$, by the complete invariance of $J(f)$, the continuity of f, and the fact that $f^{-1}(E_n) \cap \partial D = \emptyset$. Also K_{n-1} meets $\gamma_{n-1} \cap D$, by Lemma 5.1(b). Now let C be a component of K_{n-1}. Then $f(C)$ is connected, so it is a subset of a component Γ of K_n. The component of $f^{-1}(\Gamma)$ which contains C is itself contained in K_{n-1}, by Lemma 5.1(c), so we deduce that C is a component of $f^{-1}(\Gamma)$ which lies in $E_{n-1} \cap D$. Because Γ is unbounded by hypothesis and f is holomorphic in D, the set C is unbounded. This completes the proof of Lemma 5.6. □

To prove that $A(f,D) \cap J(f)$ has at least one unbounded component, we apply Lemma 5.6 repeatedly. First we may assume that the outer sequence (E_n) is chosen in such a way that each γ_n, $n = 0, 1, \ldots,$ meets an unbounded component of $J(f)$. Then, for $n = 0, 1, \ldots,$ let K_n^0 denote the union of *all* the unbounded components of $J(f) \cap E_n$. The set K_n^0 is nonempty and meets γ_n, and we can show that K_n^0 is a closed set as in [38, proof of Lemma 9].

We deduce, by Lemma 5.6, that $K_{n-1}^1 = f^{-1}(K_n^0) \cap E_{n-1} \cap D$ is a closed subset of $J(f) \cap E_{n-1} \cap D$, which meets γ_{n-1} and whose components are unbounded. Applying Lemma 5.6 repeatedly in this way we obtain, for $k = 0, 1, \ldots, n$, a closed subset K_{n-k}^k of $J(f) \cap E_{n-k} \cap D$, which meets γ_{n-k} and whose components are unbounded. Clearly

$$J(f) \cap E_0 \cap D \supset K_0^0 \supset K_0^1 \supset K_0^2 \supset \cdots .$$

Then $(K_0^n \cup \{\infty\})_{n\geq 0}$ is a nested sequence of continua in $\hat{\mathbb{C}}$, each meeting γ_0 and including ∞, whose intersection, K say, must be a continuum in $\hat{\mathbb{C}}$ meeting γ_0 and including ∞. Thus any component Γ of $\bigcap_{n=0}^{\infty} K_0^n = K \setminus \{\infty\}$ is unbounded.

But if $z \in \Gamma$, then we have $z \in K_0^{n+1}$ so $f^n(z) \in J(f) \cap E_n \cap D$, for $n = 0, 1, \ldots$. Hence $z \in A(f, D) \cap J(f)$, by Lemma 5.2. Thus $\Gamma \subset A(f, D) \cap J(f)$, so the proof of Theorem 5.3 is complete.

6. Functions with a logarithmic tract

Let f be a function with a direct tract D. We consider the set

$$A'(f, D) = \{z \in D : \text{ there exists } L \in \mathbb{N} \text{ such that } f^n(z) \in D \text{ and } |f^{n+L}(z)| \geq M_D^n(\rho) \text{ for all } n \in \mathbb{N}\},$$

whose definition does not require that f is defined outside of D. As with the set $A(f, D)$ that we considered earlier, the set $A'(f, D)$ does not depend on ρ, as long as ρ is chosen to be sufficiently large. The reason that we considered the set $A(f, D)$ earlier (rather than the set $A'(f, D)$) was that, for an entire function, all the components of $A(f, D)$ are unbounded whereas the set $A'(f, D)$ may have bounded components; we give an example of such a function at the end of this section. In the current situation, however, the function f may not be defined outside of D and so it makes sense to consider the set $A'(f, D)$. Note that, for a meromorphic function f with a direct tract D and for ρ large enough, we have

$$A(f, D, \rho) \subset A'(f, D) \subset A(f, D) \subset I(f). \tag{6.1}$$

The main results of this section give estimates on the size of the set $A'(f, D)$ when D is a logarithmic tract of f.

Theorem 6.1. *Let D be a logarithmic tract of f. Then the upper box dimension of $A'(f, D)$ is equal to* 2.

Theorem 6.2. *Let D be a logarithmic tract of f. Then the Hausdorff dimension of $\overline{A'(f, D)}$ is strictly greater than* 1.

We also note that the following result can be deduced from Lemma 6.4 below as in the proof of [34, Theorem A].

Theorem 6.3. *Let f be a transcendental meromorphic function with a logarithmic tract D. Then $A(f, D) \subset J(f)$.*

It follows from Theorem 6.1, Theorem 6.3 and (6.1) that if f is a transcendental meromorphic function with a logarithmic tract then the upper box dimension of $I(f) \cap J(f)$ (and so that of $J(f)$) is equal to 2. Hence, by [37, Theorem 1.2], the packing dimension of $J(f)$ is also equal to 2. This proves Theorem 1.4. On the other hand, Theorem 1.5 follows from Theorem 6.2, Theorem 6.3 and the fact that $J(f)$ is closed.

While our proof of Theorem 1.5 is based on the set of escaping points, Baranski, Karpinska and Zdunik [3] have recently shown that the set of points in the Julia set whose orbit is bounded and contained in a logarithmic tract also has Hausdorff dimension greater than 1, thereby giving an alternative proof of Theorem 1.5.

We now state a number of properties of functions with a logarithmic tract. Most of these results can be proved in a similar way to the analogous results for transcendental meromorphic functions in the class B. We then explain how Theorem 6.1 and Theorem 6.2 can be proved by making small changes to the proofs for meromorphic functions with finitely many poles in the class B.

First we note the following result.

Lemma 6.1. *Let D be a logarithmic tract of f. Then D is simply connected. If $|f(z)| = R$ for $z \in \partial D$ and $r > R$, then $\{z \in D : |f(z)| > r\}$ is a logarithmic tract of f, which is bounded by an unbounded simple analytic curve.*

The following result can be deduced from Koebe's distortion theorem as in [45, Lemma 2.3] by taking a suitable cover of the annulus by discs. Here $L = 81$.

Lemma 6.2. *Let D be a logarithmic tract of f with $|f(z)| = R$ for $z \in \partial D$. Let $|w_0| = r \geq 2R$ and let g be a branch of f^{-1} defined at w_0 such that $g(w_0) \in D$. If g is analytically continued along a curve γ that winds at most once round 0 and lies within $\{w \in \mathbb{C} : 4r/5 \leq |w| \leq 5r/4\}$, then*

$$|g'(w_0)|/L^{26} \leq |g'(w)| \leq L^{26}|g'(w_0)|,$$

for each $w \in \gamma$.

The next result can be proved in a similar way to the analogous result for meromorphic functions in the class B by using Lemma 6.1; see [33, Lemma 2.2], for example.

Lemma 6.3. *Let D be a logarithmic tract of f. There exists $R_1(f) > 0$ such that if $z \in D$ and $|z|, |f(z)| > R_1(f)$, then*

$$|f'(z)| > \frac{|f(z)| \log |f(z)|}{16\pi|z|}.$$

By repeatedly applying Lemma 6.3, we obtain the following.

Lemma 6.4. *Let D be a logarithmic tract of f and let $n \in \mathbb{N}$. If $f^k(z) \in D$ and $|f^k(z)| > \max\{R_1(f), e^{16\pi}\}$, for $0 \leq k \leq n$, then*

$$|(f^n)'(z)| > \frac{|f^n(z)| \log |f^n(z)|}{16\pi|z|}.$$

Using Koebe's distortion theorem together with Lemma 6.3, the following result can be proved in the same way as the analogous result for entire functions in the class B; see [45, Lemma 2.6].

Lemma 6.5. *Let D be a logarithmic tract of f. There exists $R_2(f) \geq R_1(f)$ such that, if $f^k(z) \in D$ for $0 \leq k < n$ and $|f^k(z)| \geq R_2(f)$ for $0 \leq k \leq n$, then the branch g of f^{-n} that maps $f^n(z)$ to z is defined on $D(f^n(z), |f^n(z)|/4)$ and satisfies, for each $0 \leq k < n$:*

(1) *for each $K \geq 4$,*

$$f^k g(D(f^n(z), |f^n(z)|/K)) \subset D(f^k(z), |f^k(z)|/(4K));$$

(2) *$f^k \circ g$ is univalent in $D(f^n(z), |f^n(z)|/4)$;*

(3) *if $w \in D(f^n(z), |f^n(z)|/8)$ then*

$$|(f^k \circ g)'(f^n(z))|/L \leq |(f^k \circ g)'(w)| \leq L|(f^k \circ g)'(f^n(z))|.$$

The next lemma gives information on the growth in a logarithmic tract. It can be proved in the same way as the analogous result for entire functions in the class B; see [37, Lemma 3.5] and also the similar argument we give at the end of Section 9.

Lemma 6.6. *Let D be a logarithmic tract of f. There exist constants $c = c(f) > 0$ and $R_3 = R_3(f) > 1$ such that*

$$\log M_D(r) \geq cr^{1/2} \geq (\log 2r)^2, \quad \textit{for } r \geq R_3.$$

We use the following result in the proof of Theorem 6.2; this requires a slightly different proof to that given for the analogous result in [39]. We use $C(r)$ to denote the circle $\{z \in \mathbb{C} : |z| = r\}$.

Lemma 6.7. *Let D be a logarithmic tract of f. There exists $R_4(f) > 0$ such that, if $r \geq R_4(f)$, then there is an unbounded simple analytic curve $\Gamma \subset D$ with $|f(z)| = r$ on Γ and $\Gamma \cap C(r) \neq \emptyset$.*

Proof. If r is sufficiently large, then $M_D(r) > r$. For such r, we take a point $z \in D \cap C(r)$ for which $|f(z)| = M_D(r)$ and let D_r denote the component of $f^{-1}(\mathbb{C} \setminus \overline{D}(0, r))$ which contains z. Lemma 6.1 implies that D_r is a logarithmic tract whose boundary Γ is an unbounded simple analytic curve on which $|f(z)| = r$. We have seen that $D_r \cap C(r) \neq \emptyset$ and, since D_r is simply connected, it follows that $\Gamma \cap C(r) \neq \emptyset$. □

The following result will also be used in the proof of Theorem 6.2. This can be proved in the same way as the analogous result for meromorphic functions with finitely many poles in class B; see [39, Lemma 2.9]. (Note that by using Lemma 6.7 we are able to ensure that the pre-images of the annulus lie in the tract D.)

Lemma 6.8. *Let D be a logarithmic tract of f. There exists $R_5(f) > 0$ such that, if $|w_0| = r \geq R_5(f)$, then there exists z_0 such that*

1. *$f(z_0) = w_0$;*
2. *$299r/300 \leq |z_0| \leq 301r/300$;*
3. *$D(z_0, r/100) \cap D$ contains at least two pre-images under f of each point in $\{w \in \mathbb{C} : 3r/4 < |w| < 5r/4\}$.*

In order to prove Theorem 6.1, we also need an estimate on the growth of the function $a(r, v)$ which was defined in (2.2).

To do this we use the following well-known lemma about real functions, whose short proof we include for completeness.

Lemma 6.9. *Let $x_0, \varepsilon > 0$ and let $h, H : [x_0, \infty) \to (0, \infty)$, with h increasing and $H(x) = \int_{x_0}^{x} h(s)ds + H(x_0)$. Then there exists a set $E \subset [x_0, \infty)$ of finite measure such that*

$$h(x) \leq H(x)^{1+\varepsilon}, \tag{6.2}$$

for $x \geq x_0$, $x \notin E$.

Proof. Defining E as the set of all $x \geq x_0$ where (6.2) does not hold, we have

$$\int_E dx \leq \int_E \frac{h(x)}{H(x)^{1+\varepsilon}}\, dx \leq \int_{x_0}^{\infty} \frac{h(x)}{H(x)^{1+\varepsilon}}\, dx = \int_{H(x_0)}^{\infty} \frac{1}{u^{1+\varepsilon}}\, du < \infty$$

and the conclusion follows. □

For a subharmonic function v we can take $h(x) = a(e^x, v)$ and $H(x) = B(e^x, v)$ in Lemma 6.9. With $r = e^x$ this yields the following result.

Lemma 6.10. *Let $v : \mathbb{C} \to [-\infty, \infty)$ be subharmonic and let $\varepsilon > 0$. Then there exists a set $F \subset [1, \infty)$ of finite logarithmic measure such that*

$$a(r, v) \leq B(r, v)^{1+\varepsilon}, \tag{6.3}$$

for $r \geq 1$, $r \notin F$.

We are now in a position to show how Theorem 6.1 can be proved by modifying the proof of [37, Theorem 1.1]. The following changes are needed for this proof to be applied to a function f with a logarithmic tract D instead of to an entire function in the class B.

1. In [37] we showed that the upper box dimension of the set $A(f)$ is equal to 2 when f is an entire function in the class B. The set $A(f)$ should be replaced by the set $A'(f, D)$ throughout. The proof of [37, Theorem 1.1] uses the fact that the set $A(f)$ contains an unbounded closed connected set — Theorem 3.3 and (6.1) show that the set $A'(f, D)$ also has this property.

2. In [37, Section 3] we gave several properties of entire functions in the class B. These properties also hold for functions with a logarithmic tract; see Lemmas 6.3, 6.4, 6.5 and 6.6.

3. The argument in [37, Section 5] develops the method of Eremenko based on Wiman-Valiron theory in order to construct a fast escaping orbit $(f^n(z_0))$ such that $|f^{n+1}(z_0)| \sim M(|f^n(z_0)|, f)$. The Wiman-Valiron estimates (5.3)-(5.5) in [37] can be replaced by (2.8) and (2.10). Throughout the argument, the central index $N(r)$ should be replaced by $a(r, v)$ and the maximum modulus $M(r, f)$ should be replaced by $M_D(r)$ to give an orbit $f^n(z_0)$ lying in D such that $|f^{n+1}(z_0)| \sim M_D(|f^n(z_0)|)$. In [37, Section 5], Lemma 5.1 is used to establish (5.10) and (5.11) which are then used to establish (5.16). It follows from (2.6) and Lemma 6.10 that the sequence (r_n) can be chosen so that the inequalities of (5.16) hold with $N(r)$ replaced by $a(r, v)$. (As in [37], there is plenty of freedom in choosing the sequence (r_n) and so a set of finite logarithmic measure can be avoided.)

3. In the current context, the inverse function G which appears in the proof in [37] is defined in a similar way in relation to points on the orbit $(f^n(z_0))$, but now its image values lie entirely in the tract.

We now show how Theorem 6.2 can be proved by modifying the proof of [39, Theorem 3]. The following changes are needed for this proof to be applied to a function f with a logarithmic tract D instead of to a meromorphic function with finitely many poles in the class B.

1. In [39, Section 2] we gave several properties of functions with finitely many poles in the class B. The analogous results for functions with a logarithmic tract were stated at the beginning of this section.

2. In [39, Section 3] we studied a set I consisting of certain pre-images of a point z_0 in $J(f)$. Here we insist (as we may) that all these pre-images lie in the logarithmic tract D. Also, we choose the point z_0 to lie in $A'(f, D)$.

3. In [39, Lemma 3.1], for each $n \in \mathbb{N}$, we constructed 2^n simple curves $\gamma_{n,i}$. By Lemma 6.7 and Lemma 6.8, we can choose each such curve so that it begins at a point in D. Since the image of the curve lies outside a large disc, it follows that the whole curve lies in D. As there are no poles in D and the image of the curve is unbounded, the curve must join a point in D to infinity as required.

Remark. As mentioned at the beginning of this section, there exists an entire function with a direct tract D such that $A'(f, D)$ has a bounded component. An example with this property is given by

$$f(z) = e^{z+2} - e - 1$$

and $D = \{z \in \mathbb{C} : |f(z)| > R\}$ if $R > 0$ is sufficiently small. We note that D is connected and hence a tract. It follows that the sets $A(f, D)$ and $A(f)$, which were introduced in Section 3, are equal and have no bounded components. To see that $A'(f, D)$ has bounded components we note that $f(-1) = -1$ while $f(x) > x$ for $x > -1$. It is not difficult to deduce from this that $(-1, \infty) \subset A(f, D)$. In fact, noting that f is conjugate to the function $z \mapsto \lambda e^z$ for $\lambda = e^{1-e}$ it can be deduced from the results in [8, 9] that $(-1, \infty)$ is a component of $A(f, D)$.

Now f has a zero $\xi \in (-1, 0)$ and there exists a unique $\eta \in (\xi, \infty)$ with $f(\eta) = R$. It follows that $(\eta, \infty) \subset A'(f, D)$, and since $A'(f, D) \subset A(f, D)$ this implies that (η, ∞) is a component of $A'(f, D)$. However, if R is small enough then there exists $x \in D \cap (-1, \xi)$ satisfying $f(x) \in (\eta, \infty)$ and $|f(x)| > R$. It follows that $x \in A'(f, D)$, but the component of $A'(f, D)$ which contains x is contained in $(-1, \xi)$ and hence is bounded.

7. Results of Teichmüller and Selberg

It seems reasonable to expect that the dynamics of a meromorphic function with "few poles" are similar to that of an entire function. (In fact it was this question that motivated our work.) We will state two results of Teichmüller and Selberg which imply that this is indeed the case under suitable additional hypotheses.

Teichmüller's result [47] says that a meromorphic function in class B has a logarithmic singularity if it has few poles and if the multiplicity of the poles is uniformly bounded. Using the standard terminology of Nevanlinna theory [17, 28], Teichmüller's result can be stated as follows.

Proposition 7.1. *Let f be a transcendental meromorphic function in the class B, and suppose that there exists $N \in \mathbb{N}$ such that the poles of f have multiplicity at most N. If $\delta(\infty, f) > 0$ or, more generally, if $m(r, f)$ is unbounded, then f has a logarithmic singularity over infinity.*

Teichmüller states this only for the case where the singularities of the inverse function of f lie over finitely many points. (This class of functions is usually denoted by S today.) However, his proof extends to class B without change.

Selberg [44] gave a variation of Teichmüller's result which even includes some functions that are not in the class B. His proof yields the following result.

Proposition 7.2. *Let f be a transcendental meromorphic function. Suppose that there exist $R > 0$ and $N \in \mathbb{N}$ such that for each component U of $f^{-1}(\widehat{\mathbb{C}} \setminus \overline{D}(0, R))$ which contains a pole the map $f : U \to \widehat{\mathbb{C}} \setminus \overline{D}(0, R))$ is a proper map of degree at most N. If $\delta(\infty, f) > 0$, or more generally, if $m(r, f)/\log r$ is unbounded, then f has a direct singularity over infinity.*

Note that functions in class B whose poles have multiplicity at most N satisfy the hypothesis of Selberg's result. So a slightly weakened form of Teichmüller's result follows from Selberg's result: the hypothesis that $m(r, f)$ is unbounded has to be replaced by the hypothesis that $m(r, f)/\log r$ is unbounded.

8. Examples

First we remark that it is easy to find a transcendental entire function which has a logarithmic tract but is not in class B. For example, the functions $f(z) = \exp(z^2)\cos(\sqrt{z})$ and $f(z) = \exp(z^2)/\Gamma(-z)$ are not in class B but each has a logarithmic tract which includes most of the negative real axis and a direct (non-logarithmic) tract which includes most of the positive real axis (apart from gaps near the the zeros of f).

Here we give several examples, each with a direct tract and infinitely many poles.

Example 8.1. Let $\lambda \in \mathbb{C} \setminus \{0\}$ and

$$f(z) = \lambda \frac{e^{2z} - 1}{e^z - 1/z}.$$

Then $|f(z)| \le |\lambda|(e^2 + 1)/(e - 1)$ for $\operatorname{Re} z = 1$ while $|f(x)| \to \infty$ as $x \to \pm\infty$ for $x \in \mathbb{R}$. For $R > |\lambda|(e^2 + 1)/(e - 1)$ we thus find that the set $f^{-1}(\widehat{\mathbb{C}} \setminus \overline{D}(0, R))$ has at least two unbounded components, one containing the large positive numbers and one containing the large negative numbers. The component containing the large positive numbers is a direct tract, while the one containing the large negative numbers is not. Now, f is bounded on each line $\{z \in \mathbb{C} : \operatorname{Re} z = a\}$ for $a \ge 1$ and it is not difficult to check that f' has no zeros in $\{z \in \mathbb{C} : \operatorname{Re} z \ge a\}$ for large enough $a > 0$. Also f has no finite asymptotic values. Thus, for large enough $R > 0$ the direct tract containing the large positive numbers is actually a logarithmic tract.

For all $\lambda \in \mathbb{C} \setminus \{0\}$, the function f has a superattracting fixed point at 0. For $\lambda > 1$ the function has a Baker domain containing all large negative numbers. For $\lambda = 1$ there are infinitely many Baker domains in the left half-plane; see [33, Theorem 1]. Figure 1 shows the Julia and Fatou sets for the parameters $\lambda = \frac{1}{2}$, $\lambda = 1$ and $\lambda = 2$ (from left to right). The attracting basin of zero is drawn grey, the points tending to infinity in the logarithmic tract are black and the remaining points (including those in the Baker domains) are white. The range shown is given by $-10 \le \operatorname{Re} z \le 8$, $|\operatorname{Im} z| \le 12$.

Example 8.2. It follows from Stirling's formula that the gamma function Γ has a direct tract in the right half-plane. Of course, this also holds for $\Gamma(z + a)$ with $a \in \mathbb{C}$. One can also show

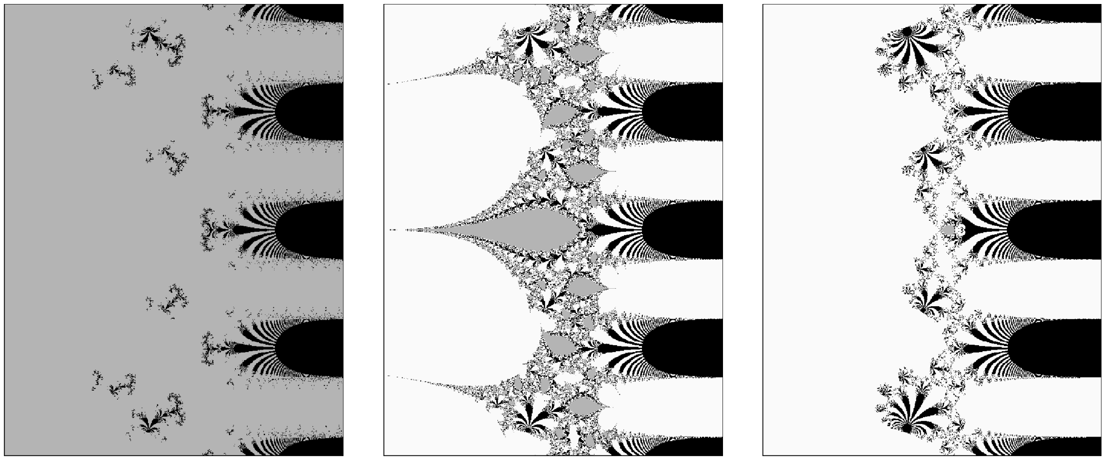

FIGURE 1. The Julia sets of the functions from Example 8.1.

that $\Gamma(z)$ and hence the functions $\Gamma(z+a)$ are in class B and thus the direct tract is actually logarithmic. We note that the gamma function also illustrates the results of Section 7, since $\delta(\infty, \Gamma) = 1$.

The function $\Gamma(z+1)$ has an attracting fixed point at 1. The point 0 is in the basin of attraction of this fixed point, and since the gamma function is close to 0 in the left half-plane except for small neighbourhoods of the poles, large parts of the left half-plane are also contained in the attracting basin of 1.

The function $\Gamma(z)$ also has an attracting fixed point at 1. However, 0 is not in the basin of attraction, but 0 is a pole. This explains the rather complicated structure of the Julia set in the left half-plane, which is mapped into a neighbourhood of 0.

Figure 2 shows the Julia and Fatou sets of $\Gamma(z)$ (left) and $\Gamma(z+1)$ (middle). The attracting basin of the fixed point at 1 is drawn white, the points tending to infinity in the logarithmic tract are black and the remaining points are grey. The range shown is given by $-5 \leq \operatorname{Re} z \leq 10$, $|\operatorname{Im} z| \leq 10$.

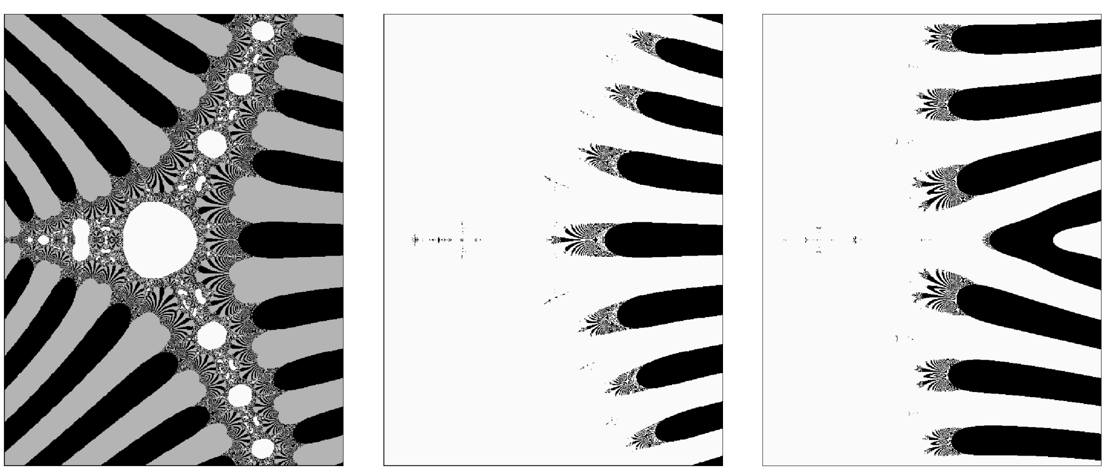

FIGURE 2. The Julia sets of $\Gamma(z)$, $\Gamma(z+1)$ and $\Gamma(z+1)\cos(z)$.

Example 8.3. Stirling's formula yields that the function $f(z) = \Gamma(z+1)\cos(z)$ is bounded on the imaginary axis and that it is unbounded in the right half-plane. Thus f has a direct tract. We remark that f satisfies the hypotheses of Proposition 7.2.

We note that the tract of f is not logarithmic. This can be shown directly, but can also be deduced from general principles. For example, this follows since a hypothetical logarithmic tract would be simply connected. By symmetry and since the zeros of the cosine are not in a tract, the function f would then have two logarithmic tracts in the right half-plane. On the other hand, f has order one and hence it follows from the Denjoy-Carleman-Ahlfors Theorem [28, Section XI.4] that f can have at most one direct tract in a half-plane.

The Julia and Fatou sets of f are shown in the right picture of Figure 2. The function has the attracting fixed point $0.6965\ldots$, whose attracting basin is drawn white, while the points tending to infinity in the direct tract are black. The range shown is given by $-5 \le \operatorname{Re} z \le 10$, $|\operatorname{Im} z| \le 10$.

9. Proof of Theorem 2.1

We denote the Riesz measure of a subharmonic function u by μ_u; see [20, Section 3.5].

Lemma 9.1. *Let $u : \mathbb{C} \to [-\infty, \infty)$ be subharmonic and let γ be a closed, piecewise smooth Jordan curve. Denote the interior of γ by $\operatorname{int}(\gamma)$. Suppose that γ has a neighbourhood U where u has the form $u = \log |f|$ for some nonvanishing holomorphic function $f : U \to \mathbb{C}$. Then $\mu_u(\operatorname{int}(\gamma))$ is a nonnegative integer.*

Proof. For $z \in U \cup \operatorname{int}(\gamma)$ we have (see [20, Theorem 3.9])

$$u(z) = \int_E \log |z - \zeta| d\mu_u(\zeta) + h(z),$$

where $E = \operatorname{supp}(\mu_u) \cap \operatorname{int}(\gamma)$ is compact and h is harmonic in $U \cup \operatorname{int}(\gamma)$. We may assume that $U \cup \operatorname{int}(\gamma)$ is simply connected. Differentiating we find that if $z \in U$, then

$$\frac{f'(z)}{f(z)} = \int_E \frac{1}{z - \zeta} d\mu_u(\zeta) + g(z)$$

where g is holomorphic in $U \cup \operatorname{int}(\gamma)$. Integrating along γ yields

$$\int_\gamma \frac{f'(z)}{f(z)} dz = \int_E \left(\int_\gamma \frac{dz}{z - \zeta} \right) d\mu_u(\zeta) = 2\pi i \int_E d\mu_u(\zeta) = 2\pi i \mu_u(E) = 2\pi i \mu_u(\operatorname{int}(\gamma)).$$

The integral on the left is purely imaginary, and its imaginary part is the increase of the argument of $\log f(z)$ as z traverses the curve γ. Clearly, this increase is a multiple of 2π. The conclusion follows. □

An immediate consequence of Lemma 9.1 is the following result.

Lemma 9.2. *Let D be a direct tract of f and let v be defined by (2.3). Let K be a bounded component of the complement of D. Then $\mu_v(K) \ge 1$.*

Proof. It follows from Lemma 9.1 that $\mu_v(K)$ is a nonnegative integer, and the case $\mu_v(K) = 0$ is excluded since v is not harmonic on the boundary of K. □

The following lemma is essentially Jensen's formula for subharmonic functions; see [20, Section 3.9].

Lemma 9.3. *Let $u : \mathbb{C} \to [-\infty, \infty)$ be subharmonic, $a \in \mathbb{C}$ and $r > 0$. For $t > 0$ put $n(a, t, u) = \mu_u(\overline{D}(a, t))$. Then*

$$\frac{1}{2\pi} \int_0^{2\pi} u(a + re^{i\varphi}) d\varphi = \int_0^r \frac{n(a, t, u)}{t} dt + u(a).$$

In particular, taking $a = 0$ we obtain a lower bound for $B(r, u)$ from Lemma 9.3. Another lower bound is given by the following result, which in slightly different form can be found in [19, p. 548] or [48, p. 117]. To state it we fix a domain D and, for a circle $C(a, r) = \{z \in \mathbb{C} : |z - a| = r\}$ which intersects D, we denote by $r\theta(a, r)$ the linear measure of the intersection. Let

$\theta^*(a,r) = \theta(a,r)$ if $C(a,r) \not\subset D$ and let $\theta^*(a,r) = \infty$, and thus $1/\theta^*(a,r) = 0$, if $C(a,r) \subset D$. In this lemma, we use the more general definition

$$B(r,u) = \max_{|z|=r, z\in D} u(z),$$

since we do not assume that u is defined in $\mathbb{C}$.

Lemma 9.4. *Let $D \subset \mathbb{C}$ be an unbounded domain and suppose that $u : \overline{D} \to [-\infty, \infty)$ is continuous in $\overline{D}$ and subharmonic in D. Suppose also that u is bounded above on ∂D, but not bounded above in D. Let $0 < \kappa < 1$ and let $r_0 > 0$ be such that $C(0, r_0)$ intersects D. Then*

$$\log B(r,u) \geq \pi \int_{r_0}^{\kappa r} \frac{dt}{t\theta^*(0,t)} - O(1)$$

as $r \to \infty$.

Proof of Theorem 2.1. As already mentioned after the statement of Theorem 2.1, (2.5) follows from (2.4). Thus it suffices to prove (2.4).

Suppose first that the complement of D has infinitely many bounded components. It then follows from Lemma 9.2 that the function $n(0,t,v)$ defined in Lemma 9.3 tends to ∞ as t tends to ∞. By Lemma 9.3 we have

$$B(r,v) \geq \frac{1}{2\pi} \int_0^{2\pi} v(re^{i\varphi}) d\varphi = \int_0^r \frac{n(0,t,v)}{t} dt + v(0) \geq \int_{\sqrt{r}}^r \frac{n(0,t,v)}{t} dt \geq \tfrac{1}{2} n(0, \sqrt{r}, v) \log r,$$

and (2.4) follows.

Suppose now that the complement of D has only finitely many bounded components. Since the complement of D is unbounded this implies that the complement of D has at least one unbounded component. For the function θ^* defined before Lemma 9.4 we thus have $\theta^*(0,t) \leq 2\pi$ for all large t, say for $t \geq t_0 > 1$. Lemma 9.4 now yields

$$\log B(r,v) \geq \pi \int_{t_0}^{\kappa r} \frac{dt}{t\theta^*(0,t)} - O(1) \geq \tfrac{1}{2} \log r - O(1)$$

and hence

$$B(r,v) \geq c\sqrt{r}$$

for some $c > 0$ and all large r. Again (2.4) follows. □

10. Some estimates of harmonic measure

In this section and the next we give some auxiliary results that are needed in the proof of Theorem 2.2. The following estimate of harmonic measure can be found in [48, p. 112]. We mention that Lemma 9.4 is deduced from this estimate in [48]. We use the standard notation for harmonic measure and the terminology introduced before Lemma 9.4.

Lemma 10.1. *Let $D \subset \mathbb{C}$ be a domain, $a \in D$ and $r > 0$. Let V be the component of $D \cap D(a,r)$ that contains a and let $\Gamma = \partial V \cap C(a,r)$. For $0 < \kappa < 1$ we then have*

$$\omega(a, \Gamma, V) \leq \frac{3}{\sqrt{1-\kappa}} \exp\left(-\pi \int_0^{\kappa r} \frac{dt}{t\theta^*(a,t)}\right).$$

The following result is known as the two constants theorem; see [28, Section III.2], although our version is somewhat different from the one given there.

Lemma 10.2. *Let V be a bounded domain with piecewise smooth boundary. Let Σ be a subset of ∂V consisting of finitely many boundary arcs and let m, M be real constants with $m < M$. Suppose that $u : \overline{V} \to [-\infty, \infty)$ is continuous in $\overline{V}$ and subharmonic in V. Suppose also that $u(z) \leq M$ for all $z \in \overline{V}$ and that $u(z) \leq m$ for $z \in \Sigma$. Then*

$$u(z) \leq \omega(z, \Sigma, V) m + (1 - \omega(z, \Sigma, V)) M.$$

To prove the lemma one has only to note that the right-hand side is a harmonic function which majorizes u on the boundary of V. Since u is subharmonic it thus majorizes u also in the interior of V.

11. Growth lemmas for real functions

The following result is a version of a classical lemma due to Borel and Nevanlinna; see [27] or [7, Section 3.3].

Lemma 11.1. *Let $x_0 > 0$ and let $T : [x_0, \infty) \to (0, \infty)$ be nondecreasing. Let $0 < \alpha < \beta$. Then there exists a set $E \subset [x_0, \infty)$ of finite measure such that if $x \notin E$, then*

$$T\left(x + T(x)^{-\beta}\right) < \left(1 + T(x)^{-\alpha}\right) T(x) \tag{11.1}$$

and

$$T\left(x - T(x)^{-\beta}\right) > \left(1 - T(x)^{-\alpha}\right) T(x). \tag{11.2}$$

Proof. Let E_1 be the subset of $[x_0, \infty)$ where (11.1) fails and let E_2 be the one where (11.2) fails, so that we can take $E = E_1 \cup E_2$. To estimate the size of E_1 we may assume that E_1 is unbounded. We choose $x_1 \in E_1 \cap [\inf E_1, \inf E_1 + \frac{1}{2}]$ and put $x_1' = x_1 + T(x_1)^{-\beta}$. Recursively we then choose

$$x_j \in E_1 \cap \left[\inf\left(E_1 \cap [x_{j-1}', \infty)\right), \inf\left(E_1 \cap [x_{j-1}', \infty)\right) + 2^{-j}\right]$$

and put $x_j' = x_j + T(x_j)^{-\beta}$. Then

$$T(x_{j+1}) \geq T(x_j') = T\left(x_j + T(x_j)^{-\beta}\right) \geq \left(1 + T(x_j)^{-\alpha}\right) T(x_j).$$

It follows from this that $T(x_j) \to \infty$, since otherwise there exists $M > 0$ with $T(x_j) \leq M$ for all $j \in \mathbb{N}$ and thus

$$M \geq T(x_{j+1}) \geq \left(1 + M^{-\alpha}\right) T(x_j) \geq \left(1 + M^{-\alpha}\right)^j T(x_1),$$

for all $j \in \mathbb{N}$, a contradiction.

We shall prove by induction that there exists a constant $c_1 > 0$ such that

$$T(x_j) \geq c_1 j^{1/\alpha} \tag{11.3}$$

for all $j \in \mathbb{N}$.

In order to do so we note first

$$\left(\frac{j+1}{j}\right)^{1/\alpha} = \left(1 + \frac{1}{j}\right)^{1/\alpha} \leq 1 + \frac{c_1^{-\alpha}}{j} \tag{11.4}$$

for all $j \in \mathbb{N}$, if c_1 is chosen small enough, say for $c_1 \leq c'$. Next we note that there exists $t_\alpha > 0$ such that the function $t \mapsto (1 + t^{-\alpha})t$ increases for $t \geq t_\alpha$.

We now choose j_0 so large that $T(x_{j_0}) \geq t_\alpha$ and $c' {j_0}^{1/\alpha} \geq t_\alpha$. Choosing c_1 such that $c_1 {j_0}^{1/\alpha} = t_\alpha$ we see that (11.3) holds for $j = j_0$. Suppose now that (11.3) holds for some $j \geq j_0$. Then

$$T(x_{j+1}) \geq \left(1 + T(x_j)^{-\alpha}\right) T(x_j) \geq \left(1 + \left(c_1 j^{1/\alpha}\right)^{-\alpha}\right) c_1 j^{1/\alpha} = \left(1 + \frac{c_1^{-\alpha}}{j}\right) c_1 j^{1/\alpha}.$$

Combining this with (11.4) we obtain

$$T(x_{j+1}) \geq c_1 (j+1)^{1/\alpha},$$

and thus we see that (11.3) holds for all $j \geq j_0$. Adjusting the value of c_1 if necessary we may thus assume that (11.3) is satisfied for all $j \in \mathbb{N}$.

Since $T(x_j) \to \infty$ and thus $x_j \to \infty$ as $j \to \infty$ we have

$$E_1 \subset \bigcup_{j=1}^{\infty} \left[x_j - 2^{-j}, x_j'\right]$$

and hence

$$\text{meas}\, E_1 \le \sum_{j=1}^{\infty} \left(x_j' - x_j + 2^{-j}\right) = \sum_{j=1}^{\infty} \frac{1}{T(x_j)^\beta} + \sum_{j=1}^{\infty} \frac{1}{2^j} \le c_1^{-\beta} \sum_{j=1}^{\infty} \frac{1}{j^{\beta/\alpha}} + 1 < \infty.$$

To estimate E_2 we proceed similarly. We may assume that $E_2 \ne \emptyset$ and fix $R > x_0$ so large that $E_2 \cap [x_0, R] \ne \emptyset$. We choose

$$z_1 \in E_2 \cap \left[\sup\left(E_2 \cap [x_0, R]\right) - \tfrac{1}{2}, \sup\left(E_2 \cap [x_0, R]\right)\right]$$

and put $z_1' = z_1 - T(z_1)^{-\beta}$. Recursively we then choose

$$z_j \in E_2 \cap \left[\sup\left(E_2 \cap [x_0, z_{j-1}']\right) - 2^{-j}, \sup\left(E_2 \cap [x_0, z_{j-1}']\right)\right]$$

and put $z_j' = z_j - T(z_j)^{-\beta}$, as long as $E_2 \cap [x_0, z_{j-1}'] \ne \emptyset$. However, since

$$\begin{aligned} T(z_{j+1}) &\le T(z_j') \\ &= T\left(z_j - T(z_j)^{-\beta}\right) \\ &\le \left(1 - T(z_j)^{-\alpha}\right) T(z_j) \\ &\le \left(1 - T(z_1)^{-\alpha}\right) T(z_j) \\ &\le \left(1 - T(z_1)^{-\alpha}\right)^j T(z_1) \end{aligned}$$

we see that the process stops and we obtain two finite sequences $(z_1, \dots, z_N)$ and $(z_1', \dots, z_N')$ with

$$E_2 \cap [x_0, R] \subset \bigcup_{j=1}^{N} [z_j', z_j + 2^{-j}].$$

With $y_j = z_{N-j+1}$ we thus have

$$E_2 \cap [x_0, R] \subset \bigcup_{j=1}^{N} [y_j', y_j + 2^{j-N-1}]$$

and

$$T(y_j) \le \left(1 - T(y_{j+1})^{-\alpha}\right) T(y_{j+1}).$$

As in the proof of (11.3) we see that for sufficiently small c_2 and large j the conditions $T(y_j) \ge c_2 j^{1/\alpha}$ and $T(y_{j+1}) < c_2 (j+1)^{1/\alpha}$ are incompatible for large j, and as before we can deduce that there exists a positive constant c_2 independent of R such that

$$T(y_j) \ge c_2 j^{1/\alpha}$$

for $1 \le j \le N$. We conclude that

$$\text{meas}\left(E_2 \cap [x_0, R]\right) \le \sum_{j=1}^{N} (y_j - y_j' + 2^{j-N-1}) = \sum_{j=1}^{N} \frac{1}{T(y_j)^\beta} + \sum_{j=1}^{N} 2^{j-N-1} \le c_2^{-\beta} \sum_{j=1}^{\infty} \frac{1}{j^{\beta/\alpha}} + 1,$$

and thus meas $E_2 < \infty$. □

We will apply this lemma to the derivative of a convex function Φ. We note here that a convex function has one-sided derivatives at all points and is differentiable except for at most countably many points. In the following result it will be irrelevant how we define the derivative at these countably many points, but to be definite we denote by Φ' the derivative from the right, which is then an increasing function.

Lemma 11.2. *Let $x_0 > 0$ and let $\Phi : [x_0, \infty) \to (0, \infty)$ be increasing and convex. Let $\beta > \frac{1}{2}$. Then there exists a set $E \subset [x_0, \infty)$ of finite measure such that*

$$\Phi(x+h) \leq \Phi(x) + \Phi'(x)h + o(1) \quad \text{for} \quad |h| \leq \Phi'(x)^{-\beta}, \quad x \notin E, \tag{11.5}$$

uniformly as $x \to \infty$.

For differentiable Φ and for h in the range $|h| \leq C/\Phi'(x)$ with a constant $C > 0$ this result was proved in [4, Lemma 2].

Proof of Lemma 11.2. Since Φ' is nondecreasing, $\lim_{x\to\infty} \Phi'(x)$ exists. It is easy to see that (11.5) holds for all large x if this limit is finite. Hence we assume that $\lim_{x\to\infty} \Phi'(x) = \infty$.

We apply Lemma 11.1 with $T = \Phi'$ and some α satisfying $\frac{1}{2} < \alpha < \beta$. For $x \notin E$ and $0 < h \leq \Phi'(x)^{-\beta}$ we then have

$$\begin{aligned}
\Phi(x+h) &= \Phi(x) + \int_x^{x+h} \Phi'(u)du \\
&\leq \Phi(x) + \Phi'(x+h)h \\
&\leq \Phi(x) + \Phi'(x + \Phi'(x)^{-\beta})h \\
&\leq \Phi(x) + \left(1 + \Phi'(x)^{-\alpha}\right) \Phi'(x)h \\
&= \Phi(x) + \Phi'(x)h + \Phi'(x)^{1-\alpha}h \\
&\leq \Phi(x) + \Phi'(x)h + \Phi'(x)^{1-\alpha-\beta} \\
&\leq \Phi(x) + \Phi'(x)h + o(1)
\end{aligned}$$

as $x \to \infty$. The case $-\Phi'(x)^{-\beta} \leq h < 0$ is analogous. □

We apply Lemma 11.2 to $\Phi(x) = B(e^x, v)$ where v is subharmonic and B is defined by (2.1). Then $\Phi'(x) = e^x B'(e^x, v) = a(e^x, v)$. With $r = e^x$ and $s = e^{x+h}$ we obtain

$$B(s,v) = \Phi(x+h) \leq \Phi(x) + \Phi'(x)h + o(1) = B(r,v) + a(r,v) \log \frac{s}{r} + o(1)$$

for $r \notin F = \exp E$, provided $|\log(s/r)| = |h| \leq \Phi'(x)^{-\beta} = a(r,v)^{-\beta}$.

Hence we obtain the following result.

Lemma 11.3. *Let $v : \mathbb{C} \to [-\infty, \infty)$ be subharmonic and let $\beta > \frac{1}{2}$. Then there exists a set $F \subset [1, \infty)$ of finite logarithmic measure such that*

$$B(s,v) \leq B(r,v) + a(r,v) \log \frac{s}{r} + o(1) \quad \text{for} \quad \left|\log \frac{s}{r}\right| \leq \frac{1}{a(r,v)^{\beta}}, \quad r \notin F,$$

uniformly as $r \to \infty$.

12. Proof of Theorem 2.2

We apply Lemma 11.3 for some $\beta < 1$ satisfying $\frac{1}{2} < \beta < \tau$ and we apply Lemma 6.10 for some $\varepsilon > 0$ such that $(1-\beta)(1+\varepsilon) < 1$. Let F be the union of the exceptional sets of these lemmas. We put $\rho = 2ra(r,v)^{-\tau}$ whenever r is so large that $a(r,v) \neq 0$.

We consider the function

$$u(z) = v(z) - v(z_r) - a(r,v) \log \frac{|z|}{r} = v(z) - B(r,v) - a(r,v) \log \frac{|z|}{r}. \tag{12.1}$$

For $z \in \overline{D}(z_r, 512\rho)$ we have

$$\left|\frac{z - z_r}{z_r}\right| \leq \frac{512\rho}{r} = \frac{1024}{a(r,v)^{\tau}} = o(1)$$

as $r \to \infty$ by (2.5) and thus

$$\left|\log \frac{|z|}{r}\right| = \left|\log \left|1 + \frac{z - z_r}{z_r}\right|\right| \leq 2\left|\frac{z - z_r}{z_r}\right| \leq \frac{2048}{a(r,v)^{\tau}} \leq \frac{1}{a(r,v)^{\beta}} \tag{12.2}$$

for large r. Since

$$u(z) \leq B(|z|, v) - B(r, v) - a(r, v) \log \frac{|z|}{r}$$

by the definition of u we conclude from Lemma 11.3 that

$$u(z) \leq o(1) \quad \text{for} \quad z \in \overline{D}(z_r, 512\rho), \quad r \notin F, \tag{12.3}$$

as $r \to \infty$.

We now show that $D(z_r, \rho) \subset D$ if $r \notin F$ is sufficiently large. Suppose that there exists $\xi \in D(z_r, \rho)$ with $\xi \notin D$. We denote by K the component of the complement of D that contains ξ and distinguish two cases.

Case 1. $K \not\subset D(z_r, 256\rho)$.

Then K intersects $\partial D(z_r, t)$ for $\rho \leq t \leq 256\rho$. Thus $\theta^*(z_r, t) \leq 2\pi$ for $\rho \leq t \leq 256\rho$. Let V be the component of $D \cap D(z_r, 512\rho)$ that contains z_r and let $\Gamma = \partial V \cap \partial D(z_r, 512\rho)$. Lemma 10.1 yields with $\kappa = \frac{1}{2}$ that

$$\omega(z_r, \Gamma, V) \leq 3\sqrt{2} \exp\left(-\pi \int_\rho^{256\rho} \frac{dt}{t\theta^*(z_r, t)}\right) \leq 3\sqrt{2} \exp\left(-\frac{1}{2}\int_\rho^{256\rho} \frac{dt}{t}\right) = \frac{3\sqrt{2}}{2^4} < \frac{1}{2}.$$

For $\Sigma = \partial V \setminus \Gamma$ we thus have

$$\omega(z_r, \Sigma, V) = 1 - \omega(z_r, \Gamma, V) > \frac{1}{2}. \tag{12.4}$$

For $z \in \Sigma$ we have $v(z) = 0$ and thus we deduce from (12.1), (12.2) and (6.3) that if $r \notin F$ is sufficiently large, then

$$\begin{aligned} u(z) &= -B(r, v) - a(r, v) \log \frac{|z|}{r} \\ &\leq -B(r, v) + a(r, v)^{1-\beta} \\ &\leq -B(r, v) + B(r, v)^{(1-\beta)(1+\varepsilon)} \\ &\leq -\tfrac{1}{2} B(r, v). \end{aligned}$$

By this inequality and (12.3), we can apply Lemma 10.2, for large $r \notin F$, with $m = -\frac{1}{2}B(r, v)$ and $M = 1$, and we obtain, by (12.4),

$$u(z_r) \leq -\tfrac{1}{2}\omega(z_r, \Sigma, V)B(r, v) + 1 - \omega(z_r, \Sigma, V) \leq -\tfrac{1}{4}B(r, v) + 1.$$

This is a contradiction since $u(z_r) = 0$ by the definition of u, but $B(r, v) \to \infty$ as $r \to \infty$.

Case 2. $K \subset D(z_r, 256\rho)$.

We again use the function u defined by (12.1). For large r we have $0 \notin D(z_r, 512\rho)$ so that the difference of u and v is harmonic in $D(z_r, 512\rho)$. Hence their Riesz measures in this disc coincide. Lemma 9.2 implies that the Riesz measure of K is a positive integer and thus $n(z_r, t, u) = n(z_r, t, v) \geq 1$ for $t \geq 256\rho$. Hence

$$\int_0^{512\rho} \frac{n(z_r, t, u)}{t} dt \geq \int_{256\rho}^{512\rho} \frac{dt}{t} = \log 2.$$

On the other hand, Lemma 9.3 and (12.3) yield

$$\int_0^{512\rho} \frac{n(z_r, t, u)}{t} dt = \frac{1}{2\pi} \int_0^{2\pi} u(z_r + 512\rho e^{i\varphi}) d\varphi - u(z_r) \leq o(1)$$

as $r \to \infty$, $r \notin F$. The last two inequalities yield a contradiction.

This completes the proof that $D(z_r, \rho) \subset D$ for large $r \notin F$. In order to prove (2.8) we note that since $D(z_r, \rho) \subset D$ we may define a holomorphic function $g : D(z_r, \rho) \to \mathbb{C}$ by

$$g(z) = \log \frac{f(z)}{f(z_r)} - a(r, v) \log \frac{z}{z_r} = \log \left(\frac{f(z)}{f(z_r)} \left(\frac{z_r}{z}\right)^{a(r,v)}\right),$$

with the branches of the logarithms chosen such that $g(z_r) = 0$. By the Borel-Carathéodory inequality (see, e.g., [49, p. 20]), we have

$$\max_{|z-z_r|=t} |g(z)| \leq 4 \max_{|z-z_r|=2t} \operatorname{Re} g(z)$$

for $0 < t < \rho/2$. Since $\operatorname{Re} g(z) = u(z)$ we can now deduce from (12.3) that if $z \in D(z_r, \rho/2) = D(z_r, ra(r,v)^{-\tau})$, then $g(z) \to 0$ as $r \to \infty$, $r \notin F$. This yields (2.8).

It follows from (2.8) and Lemma 11.3 that, as $r \to \infty$, $r \notin F$, we have

$$M_D(|z|) \geq |f(z)| \geq (1 - o(1)) \left| \frac{z}{z_r} \right|^{a(r,v)} |f(z_r)|$$

and

$$M_D(|z|) = \exp B(|z|, v) \leq \exp \left(B(r,v) + a(r,v) \log \frac{|z|}{r} + o(1) \right) = (1 + o(1)) \left| \frac{z}{z_r} \right|^{a(r,v)} |f(z_r)|$$

for $z \in D(z_r, ra(r,v)^{-\tau})$. The last two inequalities give (2.9).

13. Applications to differential equations

Wiman-Valiron theory has found many applications in the theory of differential equations in the complex domain, see, e.g., [22, 24, 50]. It seems plausible that (2.10) allows us to extend some of these results to meromorphic solutions with a direct singularity. We confine ourselves to giving one example of an application to differential equations. Another application of Theorem 2.2 to complex differential equations is given in [13]. In order to state our application, let f be meromorphic, $n \in \mathbb{N}$, $t_j \in \mathbb{N}_0 = \mathbb{N} \cup \{0\}$ for $j = 0, 1, 2, \ldots, n$, and put $t = (t_0, t_1, \ldots, t_n)$. Define $M_t[f]$ by

$$M_t[f](z) = f(z)^{t_0} f'(z)^{t_1} f''(z)^{t_2} \ldots f^{(n)}(z)^{t_n},$$

with the convention that $M_{(0)}[f] = 1$. We call $d(t) = t_0 + t_1 + \cdots + t_n$ the *degree* and $w(t) = t_1 + 2t_2 + \cdots + nt_n$ the *weight* of $M_t[f]$. An *algebraic differential equation* is an equation of the form

$$\sum_{t \in T} c_t M_t[f] = 0, \tag{13.1}$$

where the c_t are polynomials and T is a finite index set.

Theorem 13.1. *Let f be a transcendental meromorphic solution of* (13.1) *with a direct singularity over ∞. Let S be the set of all $s \in T$ for which $d(s) = \max_{t \in T} d(t)$. Then S has at least two elements.*

Let Λ be the set of all $\lambda \in \mathbb{N}$ for which there exists $s \in S$ satisfying $w(s) = \lambda$. Suppose that

$$\sum_{t \in S, w(t) = \lambda} c_t \neq 0$$

for all $\lambda \in \Lambda$. Then the order of f is at least $1/\max(\Lambda)$.

For an entire function f this result can be found in [50, p. 64, p. 71]. The proof based on Wiman-Valiron theory extends to our setting without difficulty, but for completeness we include the argument below.

For an entire solution f, Wiman-Valiron theory also yields an upper bound for the growth of f. In particular, an entire solution has finite order. For a meromorphic solution with a direct singularity over ∞ we still find that the order in a direct tract is bounded, but the method does not give any information about what happens outside the tract.

Proof of Theorem 13.1. Let D be a direct tract of f, let R be as in Definition 2.1 and define v by (2.3). We use (2.10) with $z = z_r$ and obtain

$$\sum_{t\in T} c_t(z_r) f(z_r)^{d(t)} \left(\frac{a(r,v)}{z_r}\right)^{w(t)} (1+\varepsilon_t(r)) = 0, \tag{13.2}$$

where $\varepsilon_t(r) \to 0$ as $r \to \infty$, $r \notin F$. By Lemma 6.10 we may assume that $a(r,v) \le B(r,v)^2 = (\log |f(z_r)|)^2$ for $r \notin F$ and this implies that as $r \to \infty$, $r \notin F$,

$$c_t(z_r) f(z_r)^{d(t)} \left(\frac{a(r,v)}{z_r}\right)^{w(t)} = o\left(c_s(z_r) f(z_r)^{d(s)} \left(\frac{a(r,v)}{z_r}\right)^{w(s)}\right)$$

if $d(t) < d(s)$. After dividing by $f(z_r)^d$ where $d = \max_{t\in T} d(t)$ the equation (13.2) thus takes the form

$$\sum_{t\in S} c_t(z_r) \left(\frac{a(r,v)}{z_r}\right)^{w(t)} (1+\varepsilon_t^*(r)) = 0, \tag{13.3}$$

where $\varepsilon_t^*(r) \to 0$ as $r \to \infty$, $r \notin F$. This implies that S has at least two elements.

By hypothesis,

$$u_\lambda = \sum_{s\in S, w(s)=\lambda} c_s \ne 0,$$

and hence there exist $b_\lambda \in \mathbb{C}\setminus\{0\}$ and $d_\lambda \in \mathbb{N}_0$ such that $u_\lambda(z) \sim b_\lambda z^{d_\lambda}$ as $z \to \infty$. Equation (13.3) now takes the form

$$\sum_{\lambda\in\Lambda} b_\lambda z_r^{d_\lambda - \lambda} a(r,v)^\lambda (1+\varepsilon_t^{**}(r)) = 0,$$

where $\varepsilon_t^{**}(r) \to 0$ as $r \to \infty$, $r \notin F$. Except for the terms $\varepsilon_t^{**}(r)$ and the exceptional set F, this is an algebraic equation for $a(r,v)$. This makes it plausible that $a(r,f)$ grows like a solution of the associated algebraic equation. It turns out that this is indeed the case. We omit the argument justifying this since it can be found in detail in [22, Hilfssatz 22.2]. It follows from the argument given there that there exists a positive real number τ and a positive rational number κ such that

$$a(r,v) \sim \tau r^\kappa \tag{13.4}$$

as $r \to \infty$. In fact, the possible values for κ can be computed from the Newton-Puiseux diagram associated to the algebraic equation. In particular it follows that $\kappa \ge 1/\max(\Lambda)$. Integrating (13.4) yields

$$B(r,v) \sim \frac{\tau}{\kappa} r^\kappa.$$

By [20, Theorem 3.19] we have

$$B(\rho,v) \le \left(\frac{r+\rho}{r-\rho}\right) \frac{1}{2\pi}\int_0^{2\pi} v(re^{i\varphi})d\varphi$$

for $0 < \rho < r$. Now

$$\frac{1}{2\pi}\int_0^{2\pi} v(re^{i\varphi})d\varphi \le m\left(r,\frac{f}{R}\right) \le T\left(r,\frac{f}{R}\right) \le T(r,f) + O(1)$$

as $\rho \to \infty$, where $m(r,f)$ and $T(r,f)$ are the usual quantities from Nevanlinna theory. Combining the last three estimates and choosing $r = 2\rho$ we obtain

$$T(r,f) \ge \frac{1}{3} B\left(\frac{r}{2},v\right) \ge (1-o(1)) \frac{\tau}{3\kappa 2^\kappa} r^\kappa$$

as $r \to \infty$. Thus the order of f is at least κ. ∎

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