BAKER DOMAINS FOR NEWTON'S METHOD

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Abstract. We show that there exists an entire function without finite asymptotic values for which the associated Newton function tends to infinity in some invariant domain. The question whether such a function exists had been raised by Douady.

1. Introduction and result

Let f be an entire function. Newton's method for finding the zeros of f consists of iterating the function

$$N(z) := z - \frac{f(z)}{f'(z)}.$$

If ξ is a zero of f, then $N(\xi) = \xi$ and $|N'(\xi)| < 1$, so there is an N-invariant domain U containing ξ in which the iterates N^k of N converge to ξ as $k \to \infty$. (Here N-invariance of U means that $N(U) \subset U$.)

There may also be N-invariant domains in which the iterates of N tend to ∞. A simple example is given by $f(z) = P(z)\exp Q(z)$ where P and Q are polynomials, with Q nonconstant. Then N is rational. Moreover, in the terminology of complex dynamics, the point at ∞ is a fixed point of N of multiplier 1, and the iterates of N tend to ∞ in the Leau petals associated to this fixed point.

If f does not have the above form, then N is transcendental; see [2] for an introduction to the iteration theory of transcendental meromorphic functions. A maximal N-invariant domain where the iterates of N tend to ∞ is called an *invariant Baker domain*.

A simple example (cf. [3]) is given by functions f for which $f(z) \sim \exp(-z^n)$ as $z \to \infty$ in some sector $|\arg z| < \varepsilon$. Then

$$N(z) = z + (1/n + o(1))z^{1-n}$$

and this implies that $N^k|_U \to \infty$ as $k \to \infty$ for some N-invariant domain U containing all sufficiently large positive real numbers. Note that $f(x) \to 0$ as $x \to +\infty$, $x \in \mathbb{R}$. Thus 0 is an asymptotic value of f, the positive real axis being an asymptotic path. Figuratively speaking one might say that Newton's method believes that there is a zero of f at $+\infty$, and thus it yields a domain U containing all sufficiently large positive real numbers such that $N^k(z) \to +\infty$ for $z \in U$ as $k \to \infty$.

The question arises whether an entire function f must always have 0 as an asymptotic value if N has an invariant Baker domain. This question was raised by A. Douady and has been brought to our attention by J. Rückert. It has been

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shown by X. Buff and J. Rückert [4] that the answer to this question is positive in situations much more general than those given above. However, we shall show that this is not always the case.

Theorem. *There exists an entire function f without finite asymptotic values such that $N(z) = z - f(z)/f'(z)$ has an invariant Baker domain.*

Moreover, f can be chosen to be of any preassigned order strictly between $\frac{1}{2}$ and 1.

We explain the basic idea of the construction. Using functions of the type introduced by S. K. Balašov [1], in §2 we construct an entire function f of order less than 1 (and in fact of any preassigned order strictly between $\frac{1}{2}$ and 1) which satisfies

$$f(z) \sim \sqrt[q]{z}$$

for some integer q and some branch of the q-th root as $z \to \infty$ in the spiralling region

$$S := \left\{ re^{ic\log r + i\theta} : r > 1, |\theta| < \theta_0 \right\},$$

where $c := \pi/\log(q-1)$ and $0 < \theta_0 < \pi$. Here the relation between c and q is such that S is invariant under $z \mapsto -pz$ where $p := q-1$. We show in §3 that

$$\frac{f'(z)}{f(z)} \sim \frac{1}{qz}$$

so that

$$N(z) = z - \frac{f(z)}{f'(z)} \sim -pz.$$

In fact, we have an explicit error estimate in this asymptotic equality, which yields that S contains an N-invariant domain in which the iterates of N tend to ∞. Hence N has an invariant Baker domain. Finally we show in §4, using the Denjoy-Carleman-Ahlfors Theorem, that f has no finite asymptotic values.

2. The construction of f

Let (a_k) be a sequence of complex numbers tending to infinity. For $r > 0$ let $n(r)$ be the number of a_k, taking account of repetition, in $|z| \le r$. Let

$$\rho := \limsup_{r\to\infty} \frac{\log n(r)}{\log r}.$$

Equivalently, ρ is the exponent of convergence of the sequence (a_k). It is well known that the canonical product Π whose zeros are the a_k has order ρ; that is,

$$\rho = \limsup_{r\to\infty} \frac{\log\log M(r,\Pi)}{\log r}$$

where $M(r,\Pi) := \max_{|z|=r} |\Pi(z)|$ is the maximum modulus. There are standard results concerning the asymptotic behavior of Π if all a_k lie on one ray and

$$n(r) \sim \Delta r^{\rho} \tag{2.1}$$

for some $\Delta > 0$ as $r \to \infty$. These results have been extended by Balašov [1] to the case where the a_k lie on a logarithmic spiral, say

$$a_k \in \left\{ re^{ic\log r} : r \ge 1 \right\}, \tag{2.2}$$

where $c > 0$. We quote only a simplified version of Balašov's result [1, Theorem 1], as this suffices for our purposes.

Lemma 1. *Let (a_k) be a sequence satisfying* (2.1) *and* (2.2). *Suppose that ρ is not an integer. Let Π be the canonical product formed with the a_k. Then*

$$\lim_{r\to\infty} \frac{\log \Pi(re^{ic\log r+i\theta})}{r^\rho} = -\frac{2\pi i\Delta \exp\left(i\rho\theta/(1+ic)\right)}{1-\exp\left(i2\pi\rho/(1+ic)\right)}$$

for $0 < \theta < 2\pi$ and a suitable branch of the logarithm, the convergence being uniform for $\varepsilon \le \theta \le 2\pi - \varepsilon$ if $\varepsilon > 0$. In particular,

$$\begin{aligned} & \lim_{r\to\infty} \frac{\log |\Pi(re^{ic\log r+i\theta})|}{r^\rho} \\ = & \; -2\pi\Delta \operatorname{Re}\left(\frac{i\exp\left(i\rho\theta/(1+ic)\right)}{1-\exp\left(i2\pi\rho/(1+ic)\right)}\right) \\ =: & \; h(\theta). \end{aligned} \tag{2.3}$$

Now let $\frac{1}{2} < \rho < 1$ and $\Delta > 0$. Choose $p \in \mathbb{N}$ such that

$$\mu := \frac{\rho}{1+c^2} := \frac{\rho}{1+(\pi/\log p)^2} > \frac{1}{2}, \tag{2.4}$$

thus defining $c := \pi/\log p$. Note that since $\frac{1}{2} < \mu < \rho < 1$ we have $c < 1$ and hence $p > \exp(\pi) > 23$. Let (a_k) be a sequence satisfying (2.1) and (2.2) and let Π be the canonical product formed with the a_k so that (2.3) holds.

A series of elementary modifications of Π will produce the function f of our theorem.

A computation shows that

$$\begin{aligned} h(0) &= -2\pi\Delta \operatorname{Re}\left(\frac{i}{1-\exp\left(i2\pi\rho/(1+ic)\right)}\right) \\ &= \frac{2\pi\Delta\exp(2\pi\mu c)}{\left|1-\exp\left(i2\pi\rho/(1+ic)\right)\right|^2}\sin(2\pi\mu). \end{aligned}$$

Since $\frac{1}{2} < \mu < 1$ we thus have $h(0) < 0$. Hence there exists $\theta_0 > 0$ such that $h(\theta) < 0$ for $|\theta| < \theta_0$. For $0 < \varepsilon < \theta_1 < \theta_0$ we thus deduce from (2.3) that there exists $\eta_0 > 0$ such that

$$\log|\Pi(re^{ic\log r+i\theta})| \le -\eta_0 r^\rho \quad \text{for } \varepsilon \le |\theta| \le \theta_1, \tag{2.5}$$

provided r is sufficiently large.

We show that an estimate of this type also holds for $|\theta| < \varepsilon$. In order to do so, we use a standard estimate which in slightly different form can be found in [6, p. 548] or [8, p. 117].

Lemma 2. *Let $D \subset \mathbb{C}$ be an unbounded domain. For $r > 0$ such that the circle $C_r := \{z \in \mathbb{C} : |z| = r\}$ intersects D, let $r\theta(r)$ be the linear measure of the intersection. Let $\theta^*(r) := \theta(r)$ if $C_r \not\subset D$ and let $\theta^*(r) := \infty$ and thus $1/\theta^*(r) := 0$ if $C_r \subset D$.*

Suppose that $u : \overline{D} \to [-\infty, \infty)$ is continuous in $\overline{D}$ and subharmonic in D. Suppose also that u is bounded above on ∂D, but not bounded above in D. Let $0 < \kappa < 1$ and let $R > 0$ be such that C_R intersects D. Then $B(r,u) := \max_{|z|=r} u(z)$ satisfies

$$\log B(r,u) \ge \pi \int_R^{\kappa r} \frac{dt}{t\theta^*(t)} - O(1)$$

as $r \to \infty$.

We may assume that ε in (2.5) is chosen such that $0 < \varepsilon < \pi/2$. We consider the spiralling domain

$$D := \left\{ re^{ic\log r + i\theta} : r > 1, |\theta| < \varepsilon \right\}$$

and the function

$$u(z) := \log|\Pi(z)| + \eta_0|z|^\rho.$$

Then u is continuous in $\overline{D}$, subharmonic in D and bounded above on ∂D. We claim that u is also bounded above in D. Otherwise, on applying Lemma 2 and noting that $\theta^*(r) = 2\varepsilon$ we find that

$$\log B(r,u) \geq \pi \int_R^{\kappa r} \frac{dt}{2\varepsilon t} - O(1) = \frac{\pi}{2\varepsilon}\log r - O(1) > \log r$$

and thus

$$\log M(r,\Pi) = B(r,u) - \eta_0 r^\rho > r - \eta_0 r^\rho > \frac{r}{2}$$

for large r. This implies that the order of Π is at least 1, a contradiction.

Thus u must be bounded above in D, and this, together with (2.5), implies that if $0 < \eta_1 < \eta_0$, then

$$\log|\Pi(re^{ic\log r + i\theta})| \leq -\eta_1 r^\rho \quad \text{for} \quad |\theta| \leq \theta_1 \tag{2.6}$$

and sufficiently large r.

Let L be the natural parametrization of the logarithmic spiral on which the a_k lie; that is, $L : [1,\infty) \to \mathbb{C}$, $L(t) = te^{ic\log t}$. Then $\Pi(L(t)) \to 0$ as $t \to \infty$ by (2.6). Thus there exists $t_0 > 1$ such that $|\Pi(L(t))| < |\Pi(L(t_0))|$ for $t > t_0$.

The function f of our theorem will now be defined as follows. We put $z_0 := L(t_0)$ and define $g_0(z) := \Pi(z + z_0)$. Next we put $q := p+1$ and $g_1(z) := g_0(z^q)$, and define $\sigma : [0,\infty) \to \mathbb{C}$ by $\sigma(t) = \sqrt[q]{L(t_0+t) - z_0}$, for some fixed branch of the root. We then define

$$g_2(z) := \int_0^z g_1(\zeta)^n d\zeta = \int_0^z \Pi(\zeta^q + z_0)^n d\zeta, \tag{2.7}$$

where $n \in \mathbb{N}$. It will follow easily that

$$a := \int_\sigma g_1(z)^n dz$$

is finite for all $n \in \mathbb{N}$, and using a result of W. K. Hayman [5, Lemma 1] we will see that $a = a(n) \neq 0$ if n is sufficiently large. For such n we then define $g_3(z) := g_2(z)/az$ and note that g_3 is of the form $g_3(z) = g_4(z^q)$ for some entire function g_4. The function claimed in the theorem is

$$f(z) := zg_4(z)^{q-1}.$$

We remark that we introduced z_0 and n only to ensure that $a \neq 0$. In a generic situation we could probably define g_2 directly via (2.7) with $z_0 = 0$ and $n = 1$.

To prove that f has the desired properties, we determine the asymptotic behavior of the g_j and f in spiralling regions similar to D. We first note from (2.6) that if $0 < \eta_2 < \eta_1$ and if $0 < \theta_2 < \theta_1$, then

$$\log|g_0(re^{ic\log r + i\theta})| \leq -\eta_2 r^\rho \quad \text{for} \quad |\theta| \leq \theta_2$$

and sufficiently large r. This implies that if $|\theta| \leq \theta_2/q$ and if r is sufficiently large, then

$$\log|g_1(re^{ic\log r + i\theta})| = \log|g_0(r^q e^{ic\log(r^q) + iq\theta})| \leq -\eta_2 r^{q\rho}.$$

With

$$S_1 := \left\{ re^{ic\log r + i\theta} : r > 1, |\theta| < \frac{\theta_2}{q} \right\}$$

we thus find that

$$|g_1(z)| \le \exp\left(-\eta_2 |z|^{q\rho}\right)$$

if $z \in S_1$ is sufficiently large. Moreover, $\sigma(t) \in S_1$ for large t, since for suitably chosen branches of the argument we have

$$\begin{aligned} \arg \sigma(t) &= \frac{1}{q} \arg(L(t_0 + t) - z_0) \\ &= \frac{1}{q} \arg L(t_0 + t) + o(1) \\ &= \frac{c}{q} \log |L(t_0 + t)| + o(1) \\ &= c \log |\sigma(t)| + o(1) \end{aligned}$$

as $t \to \infty$. From this it is straightforward to deduce that the integral defining a converges for all $n \in \mathbb{N}$. In order to show that $a \neq 0$ for large n, we use the following result of W. K. Hayman [5, Lemma 1].

Lemma 3. *Let γ be a Jordan arc in $\mathbb{C}$ which tends to ∞ in both directions and let g be holomorphic in a domain containing γ. Suppose that $\int_\gamma |g(z)||dz| < \infty$ and that $|g(z)| \to 0$ as $z \to \infty$ on γ. Suppose also that $|g(z)| \le M$ for z on γ, with equality for a single point z_1 on γ with $g'(z_1) = 0$. Suppose finally that γ cannot be deformed in a neighborhood of z_1 into a curve on which $|g(z)| < M$. Then*

$$\int_\gamma g(z)^n dz \neq 0$$

for all sufficiently large integers n.

We remark that since $g'(z_1) = 0$, the set $\{z \in \mathbb{C} : |g(z)| < M\}$ has at least two components whose boundary contains z_1. The condition that γ cannot be deformed in a neighborhood of z_1 into a curve on which $|g(z)| < M$ means that γ passes from one component of this set into another component at z_1.

We apply Lemma 3 with $g := g_1$, the curve γ parametrized as $\gamma : \mathbb{R} \to \mathbb{C}$,

$$\gamma(t) := \begin{cases} \sigma(-t) & \text{if } t \le 0, \\ \sigma^*(t) := e^{2\pi i/q}\sigma(t) & \text{if } t > 0, \end{cases}$$

and $z_1 := \gamma(0) = 0$. Since by the choice of z_0 we have

$$|g_1(\sigma^*(t))| = |g_1(\sigma(t))| = |\Pi(L(t_0 + t))| < |\Pi(L(t_0))| = |g_1(\sigma(0))|$$

for $t > 0$, it follows that $|g_1(\gamma(t))| < |g_1(z_1)|$ for $t \neq 0$. Moreover, $g_1'(z_1) = g_1'(0) = 0$, and thus the hypotheses of Lemma 3 are satisfied. Since

$$\int_\gamma g_1(z)^n dz = -\int_\sigma g_1(z)^n dz + \int_{\sigma^*} g_1(z)^n dz = \left(-1 + e^{2\pi i/q}\right) \int_\sigma g_1(z)^n dz$$

we conclude from Lemma 3 that $a = \int_\sigma g_1(z)^n dz \neq 0$ for sufficiently large values of n.

Thus $g_2(\sigma(t)) \to a$ as $t \to \infty$. More generally, $g_2(z) \to a$ as $z \to \infty$ in S_1. In fact, if $z \in S_1$ then

$$g_2(z) - a = \int_{\tau_z} g_1(\zeta)^n d\zeta$$

for any path τ_z joining z to ∞ in S_1. For large $z \in S_1$ and a suitable path τ_z we find that

$$|g_2(z) - a| \leq \int_{\tau_z} |g_1(\zeta)|^n |d\zeta| \leq \int_{\tau_z} \exp\left(-n\eta_2|\zeta|^{q\rho}\right)|d\zeta| \leq \exp\left(-\eta_3|z|^{q\rho}\right)$$

for some $\eta_3 > 0$. It follows that if $z \in S_1$ is sufficiently large, then

$$\left|g_3(z) - \frac{1}{z}\right| = \frac{|g_2(z) - a|}{|az|} \leq \frac{\exp\left(-\eta_3|z|^{q\rho}\right)}{|az|} \leq \exp\left(-\eta_3|z|^{q\rho}\right).$$

Now let

$$S_2 := \left\{re^{ic\log r + i\theta} : r > 1, |\theta| < \theta_2\right\}.$$

For $z \in S_2$ we have $\sqrt[q]{z} \in S_1$ for a suitable branch. For large $z \in S_2$ we thus find that

$$\left|g_4(z) - \frac{1}{\sqrt[q]{z}}\right| = \left|g_3(\sqrt[q]{z}) - \frac{1}{\sqrt[q]{z}}\right| \leq \exp\left(-\eta_3|z|^{\rho}\right);$$

i. e. if $z \in S_2$ is sufficiently large, then

$$\left|f(z) - \sqrt[q]{z}\right| \leq \exp\left(-\eta_4|z|^{\rho}\right) \tag{2.8}$$

for some $\eta_4 > 0$ and a suitable branch.

3. Newton's method for f

We choose θ_3 with $0 < \theta_3 < \theta_2$ and define

$$S_3 := \left\{re^{ic\log r + i\theta} : r > 1, |\theta| < \theta_3\right\}.$$

Then there exists $\delta > 0$ such that if $z \in S_3$ is sufficiently large, then the closed disk of radius $\delta|z|$ around z is contained in S_2. With $d(z) := f(z) - \sqrt[q]{z}$ we deduce from (2.8) that if $z \in S_3$ is sufficiently large, then

$$\begin{aligned}
\left|f'(z) - \frac{\sqrt[q]{z}}{qz}\right| &= |d'(z)| \\
&= \frac{1}{2\pi}\left|\int_{|\zeta - z| = \delta|z|} \frac{d(\zeta)}{(\zeta - z)^2} d\zeta\right| \\
&\leq \frac{1}{\delta|z|} \max_{|\zeta - z| = \delta|z|} |d(\zeta)| \\
&\leq \frac{1}{\delta|z|} \exp\left(-\eta_4(1-\delta)^{\rho}|z|^{\rho}\right) \\
&\leq \exp\left(-\eta_5|z|^{\rho}\right)
\end{aligned}$$

for some $\eta_5 > 0$. Combining this with (2.8) we find that if $z \in S_3$ is sufficiently large, then

$$\left|\frac{f(z)}{f'(z)} - qz\right| \leq \exp\left(-\eta_6|z|^{\rho}\right)$$

where $\eta_6 > 0$. Since $q = p + 1$ we deduce that

$$|N(z) + pz| = \left|z - \frac{f(z)}{f'(z)} + pz\right| = \left|\frac{f(z)}{f'(z)} - qz\right| \leq \exp\left(-\eta_6|z|^{\rho}\right) \tag{3.1}$$

for large $z \in S_3$. In particular,

$$\big||N(z)| - p|z|\big| \leq \exp\left(-\eta_6|z|^{\rho}\right)$$

which implies that

$$\big|\log|N(z)| - \log(p|z|)\big| \le \exp\left(-\eta_6|z|^\rho\right) \tag{3.2}$$

for large $z \in S_3$. Moreover, (3.1) yields

$$|\arg N(z) - \arg(-pz)| \le \exp\left(-\eta_6|z|^\rho\right) \tag{3.3}$$

for large $z \in S_3$. However, c was chosen such that $c \log p = \pi$, so we deduce from (3.2) and (3.3) that

$$\begin{aligned}
& \big|\arg N(z) - c\log|N(z)|\big| \\
\le\ & \big|\arg N(z) - \arg(-pz)\big| + \big|\arg(-pz) - c\log(p|z|)\big| \\
& + \big|c\log(p|z|) - c\log|N(z)|\big| \\
\le\ & \big|\arg(-pz) - c\log(p|z|)\big| + (1+c)\exp\left(-\eta_6|z|^\rho\right) \\
=\ & \big|\arg z + \pi - c\log p - c\log|z|\big| + (1+c)\exp\left(-\eta_6|z|^\rho\right) \\
=\ & \big|\arg z - c\log|z|\big| + (1+c)\exp\left(-\eta_6|z|^\rho\right) \\
\le\ & \big|\arg z - c\log|z|\big| + \frac{1}{2|z|}
\end{aligned}$$

for large $z \in S_3$. Since $p > 23$ we deduce from (3.1) that $|N(z)| > 2|z|$ if $z \in S_3$ and if $|z|$ is sufficiently large, say $|z| > r_0 > 1$. Combining this with the previous estimate, we conclude that if

$$\big|\arg z - c\log|z|\big| < \theta_3 - \frac{1}{|z|},$$

then

$$\big|\arg N(z) - c\log|N(z)|\big| < \theta_3 - \frac{1}{|z|} + \frac{1}{2|z|} = \theta_3 - \frac{1}{2|z|} < \theta_3 - \frac{1}{|N(z)|}$$

if $z \in S_3$ and if $|z|$ is large enough, say $|z| > r_1 > r_0$. This implies that

$$U := \left\{ re^{ic\log r + i\theta} : r > r_1, |\theta| < \theta_3 - \frac{1}{r} \right\}$$

is N-invariant. Since $|N(z)| \ge 2|z|$ for $z \in U$, we have $|N^k(z)| \ge 2^k|z|$ for $z \in U$ and $k \in \mathbb{N}$. Thus $N^k|_U \to \infty$ as $k \to \infty$. Hence U is contained in an invariant Baker domain of N.

4. Asymptotic values of f

Suppose that f has a finite asymptotic value, say $f(z) \to b \in \mathbb{C}$ as $z \to \infty$ along a curve Γ. The function

$$F(z) := \frac{f(z)^q}{z}$$

is entire since $f(0) = 0$. By (2.8) we have $F(z) \to 1$ as $z \to \infty$ along the logarithmic spiral L while $F(z) \to 0$ as $z \to \infty$ along Γ. Thus F has two finite asymptotic values. By the Denjoy-Carleman-Ahlfors Theorem (see [7, §XI.4.5]), F has order at least 1. On the other hand, F has the same order as f, which has been taken less than 1. This is a contradiction.

Remark. Our method will produce examples f of any preassigned non-integer order $\rho > 1$, as well as examples with more than one invariant Baker domain. We only sketch the modifications that have to be made.

We again choose ρ and p such that (2.4) is holds. The condition $\mu < 1$ need not be satisfied, and there may be several, say ℓ, intervals where $h(\theta)$ is negative and corresponding spiralling regions $S_1, \dots, S_\ell$ where $\Pi(z) \to 0$ as $z \to \infty$. It is not difficult to see that ℓ can be any given positive number. For each j, let L_j be a curve starting at 0 which outside the unit circle is a logarithmic spiral in S_j and which inside the unit circle is a straight line from 0 to the corresponding point of the unit circle. Deforming one of the curves L_j if necessary we may assume that there exists $z_0 \in \bigcup_{j=1}^{\ell} L_j$ such that $|\Pi(z_0)| > |\Pi(z)|$ for all $z \in \bigcup_{j=1}^{\ell} L_j$. Defining g_2 by (2.7) for some large n and then f as in §2, we find that $f(z) \sim c_j \sqrt[q]{z}$ for some $c_j \neq 0$ as $z \to \infty$ in S_j. As before, this means that $N(z) \sim -pz$ as $z \to \infty$ in S_j, $j = 1, \dots, \ell$. We thus obtain an entire function f for which N has ℓ invariant Baker domains. A difference occurs in the proof that f does not have finite asymptotic values. Here we cannot simply appeal to the classical Denjoy-Carleman-Ahlfors Theorem, but instead use that the function f constructed has only ℓ "tracts"; see [6, §8.3].

Balašov's result takes a different form if ρ is an integer, but it seems possible to treat this case along the same lines.

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