BAKER DOMAINS FOR NEWTON’S METHOD

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ABSTRACT. We show that there exists an entire function without finite asymptotic values for which the associated Newton function tends to infinity in some invariant domain. The question whether such a function exists had been raised by Douady.

1. Introduction and result

Let $f$ be an entire function. Newton’s method for finding the zeros of $f$ consists of iterating the function

$$N(z) := z - \frac{f(z)}{f'(z)}.$$ 

If $\xi$ is a zero of $f$, then $N(\xi) = \xi$ and $|N'(\xi)| < 1$, so there is an $N$-invariant domain $U$ containing $\xi$ in which the iterates $N^k$ of $N$ converge to $\xi$ as $k \to \infty$. (Here $N$-invariance of $U$ means that $N(U) \subset U$.)

There may also be $N$-invariant domains in which the iterates of $N$ tend to $\infty$. A simple example is given by $f(z) = P(z) \exp(Q(z))$ where $P$ and $Q$ are polynomials, with $Q$ nonconstant. Then $N$ is rational. Moreover, in the terminology of complex dynamics, the point at $\infty$ is a fixed point of $N$ of multiplier 1, and the iterates of $N$ tend to $\infty$ in the Leau petals associated to this fixed point.

If $f$ does not have the above form, then $N$ is transcendental; see [2] for an introduction to the iteration theory of transcendental meromorphic functions. A maximal $N$-invariant domain where the iterates of $N$ tend to $\infty$ is called an invariant Baker domain.

A simple example (cf. [3]) is given by functions $f$ for which $f(z) \sim \exp(-z^n)$ as $z \to \infty$ in some sector $|\arg z| < \varepsilon$. Then

$$N(z) = z + (1/n + o(1))z^{1-n}$$

and this implies that $N^k|_U \to \infty$ as $k \to \infty$ for some $N$-invariant domain $U$ containing all sufficiently large positive real numbers. Note that $f(x) \to 0$ as $x \to +\infty$, $x \in \mathbb{R}$. Thus 0 is an asymptotic value of $f$, the positive real axis being an asymptotic path. Figuratively speaking one might say that Newton’s method believes that there is a zero of $f$ at $+\infty$, and thus it yields a domain $U$ containing all sufficiently large positive real numbers such that $N^k(z) \to +\infty$ for $z \in U$ as $k \to \infty$.

The question arises whether an entire function $f$ must always have 0 as an asymptotic value if $N$ has an invariant Baker domain. This question was raised by A. Douady and has been brought to our attention by J. Rückert. It has been

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shown by X. Buff and J. Rückert [4] that the answer to this question is positive in situations much more general than those given above. However, we shall show that this is not always the case.

**Theorem.** There exists an entire function $f$ without finite asymptotic values such that $N(z) = z - f(z)/f'(z)$ has an invariant Baker domain.

Moreover, $f$ can be chosen to be of any preassigned order strictly between $\frac{1}{2}$ and 1.

We explain the basic idea of the construction. Using functions of the type introduced by S. K. Balasov [1], in §2 we construct an entire function $f$ of order less than 1 (and in fact of any preassigned order strictly between $\frac{1}{2}$ and 1) which satisfies

$$f(z) \sim \sqrt[2]{z}$$

for some integer $q$ and some branch of the $q$-th root as $z \to \infty$ in the spiralling region

$$S := \left\{ re^{ic\log r + i\theta} : r > 1, |\theta| < \theta_0 \right\},$$

where $c := \pi / \log(q-1)$ and $0 < \theta_0 < \pi$. Here the relation between $c$ and $q$ is such that $S$ is invariant under $z \mapsto -pz$ where $p := q - 1$. We show in §3 that

$$\frac{f'(z)}{f(z)} \sim \frac{1}{qz}$$

so that

$$N(z) = z - \frac{f(z)}{f'(z)} \sim -pz.$$ 

In fact, we have an explicit error estimate in this asymptotic equality, which yields that $S$ contains an $N$-invariant domain in which the iterates of $N$ tend to $\infty$. Hence $N$ has an invariant Baker domain. Finally we show in §4, using the Denjoy-Carleman-Ahlfors Theorem, that $f$ has no finite asymptotic values.

2. **The construction of $f$**

Let $(a_k)$ be a sequence of complex numbers tending to infinity. For $r > 0$ let $n(r)$ be the number of $a_k$, taking account of repetition, in $|z| \leq r$. Let

$$\rho := \limsup_{r \to \infty} \frac{\log n(r)}{\log r}.$$ 

Equivalently, $\rho$ is the exponent of convergence of the sequence $(a_k)$. It is well known that the canonical product $\Pi$ whose zeros are the $a_k$ has order $\rho$; that is,

$$\rho = \limsup_{r \to \infty} \frac{\log \log M(r, \Pi)}{\log r}$$

where $M(r, \Pi) := \max_{|z|=r} |\Pi(z)|$ is the maximum modulus. There are standard results concerning the asymptotic behavior of $\Pi$ if all $a_k$ lie on one ray and

$$n(r) \sim \Delta r^\rho$$

for some $\Delta > 0$ as $r \to \infty$. These results have been extended by Balasov [1] to the case where the $a_k$ lie on a logarithmic spiral, say

$$a_k \in \left\{ re^{ic\log r} : r \geq 1 \right\},$$

where $c > 0$. We quote only a simplified version of Balasov’s result [1, Theorem 1], as this suffices for our purposes.
Lemma 1. Let \((a_k)\) be a sequence satisfying (2.1) and (2.2). Suppose that \(\rho\) is not an integer. Let \(\Pi\) be the canonical product formed with the \(a_k\). Then

\[
\lim_{r \to \infty} \log \Pi \left( r e^{i \log r + i\theta} \right) = \frac{2\pi i \Delta \exp \left( i \rho \theta / (1 + ic) \right)}{1 - \exp \left( i 2\pi \rho / (1 + ic) \right)}
\]

for \(0 < \theta < 2\pi\) and a suitable branch of the logarithm, the convergence being uniform for \(\varepsilon \leq \theta \leq 2\pi - \varepsilon\) if \(\varepsilon > 0\). In particular,

\[
\lim_{r \to \infty} \log |\Pi \left( r e^{i \log r + i\theta} \right)| = -2\pi \Delta \Re \left( \frac{i \exp \left( i \rho \theta / (1 + ic) \right)}{1 - \exp \left( i 2\pi \rho / (1 + ic) \right)} \right) := h(\theta).
\]

Now let \(\frac{1}{2} < \rho < 1\) and \(\Delta > 0\). Choose \(p \in \mathbb{N}\) such that

\[
\mu := \frac{\rho}{1 + c^2} := \frac{\rho}{1 + (\pi / \log p)^2} > \frac{1}{2},
\]

thus defining \(c := \pi / \log p\). Note that since \(\frac{1}{2} < \mu < \rho < 1\) we have \(c < 1\) and hence \(p > \exp(\pi) > 23\). Let \((a_k)\) be a sequence satisfying (2.1) and (2.2) and let \(\Pi\) be the canonical product formed with the \(a_k\) so that (2.3) holds.

A series of elementary modifications of \(\Pi\) will produce the function \(f\) of our theorem.

A computation shows that

\[
h(0) = -2\pi \Delta \Re \left( \frac{i}{1 - \exp \left( i 2\pi \rho / (1 + ic) \right)} \right) = \frac{2\pi \Delta \exp(2\pi \mu c)}{|1 - \exp \left( i 2\pi \rho / (1 + ic) \right)|^2} \sin(2\pi \mu).
\]

Since \(\frac{1}{2} < \mu < 1\) we thus have \(h(0) < 0\). Hence there exists \(\theta_0 > 0\) such that \(h(\theta) < 0\) for \(|\theta| < \theta_0\). For \(0 < \varepsilon < \theta_1 < \theta_0\) we thus deduce from (2.3) that there exists \(\eta_0 > 0\) such that

\[
\log |\Pi \left( r e^{i \log r + i\theta} \right)| \leq -\eta_0 r^\rho \quad \text{for} \quad \varepsilon \leq |\theta| \leq \theta_1,
\]

provided \(r\) is sufficiently large.

We show that an estimate of this type also holds for \(|\theta| < \varepsilon\). In order to do so, we use a standard estimate which in slightly different form can be found in [6, p. 548] or [8, p. 117].

Lemma 2. Let \(D \subset \mathbb{C}\) be an unbounded domain. For \(r > 0\) such that the circle \(C_r := \{z \in \mathbb{C} : |z| = r\}\) intersects \(D\), let \(r\theta(r)\) be the linear measure of the intersection. Let \(\theta^*(r) := \theta(r)\) if \(C_r \not\subset D\) and let \(\theta^*(r) := \infty\) and thus \(1/\theta^*(r) := 0\) if \(C_r \subset D\).

Suppose that \(u : \overline{D} \to (-\infty, \infty)\) is continuous in \(\overline{D}\) and subharmonic in \(D\). Suppose also that \(u\) is bounded above on \(\partial D\), but not bounded above in \(D\). Let \(0 < \kappa < 1\) and let \(R > 0\) be such that \(C_R\) intersects \(D\). Then \(B(r, u) := \max_{|z|=r} u(z)\) satisfies

\[
\log B(r, u) \geq \pi \int_R^{\infty} \frac{dt}{t \theta^*(t)} - O(1)
\]
as \(r \to \infty\).
We may assume that $\varepsilon$ in (2.5) is chosen such that $0 < \varepsilon < \pi/2$. We consider the spiralling domain

$$D := \{ re^{i \epsilon \log r + i \theta} : r > 1, |\theta| < \varepsilon \}$$

and the function

$$u(z) := \log |\Pi(z)| + \eta_0 |z|^p.$$  

Then $u$ is continuous in $\overline{D}$, subharmonic in $D$ and bounded above on $\partial D$. We claim that $u$ is also bounded above in $D$. Otherwise, on applying Lemma 2 and noting that $\theta^*(r) = 2\varepsilon$ we find that

$$\log B(r, u) \geq \pi \int_0^{\pi \varepsilon} \frac{dt}{2\varepsilon t} - O(1) = \frac{\pi}{2\varepsilon} \log r - O(1) > \log r$$

and thus

$$\log M(r, \Pi) = B(r, u) - \eta_0 r^p > r - \eta_0 r^p > \frac{r}{2}$$

for large $r$. This implies that the order of $\Pi$ is at least 1, a contradiction.

Thus $u$ must be bounded above in $D$, and this, together with (2.5), implies that if $0 < \eta_1 < \eta_0$, then

$$\log |\Pi(re^{i \epsilon \log r + i \theta})| \leq -\eta_1 r^p \quad \text{for } |\theta| \leq \theta_1$$

and sufficiently large $r$.

Let $L$ be the natural parametrization of the logarithmic spiral on which the $a_k$ lie; that is, $L : [1, \infty) \to \mathbb{C}, L(t) = te^{i \epsilon \log t}$. Then $\Pi(L(t)) \to 0$ as $t \to \infty$ by (2.6). Thus there exists $t_0 > 1$ such that $|\Pi(L(t))| < |\Pi(L(t_0))|$ for $t > t_0$.

The function $f$ of our theorem will now be defined as follows. We put $z_0 := L(t_0)$ and define $g_0(z) := \Pi(z + z_0)$. Next we put $q := p + 1$ and $g_1(z) := g_0(z^q)$, and define $\sigma : [0, \infty) \to \mathbb{C}$ by $\sigma(t) = \sqrt[q]{L(t_0 + t)} - z_0$, for some fixed branch of the root.

We then define

$$g_2(z) := \int_0^z g_1(\zeta)^n d\zeta = \int_0^z \Pi(\zeta^q + z_0)^n d\zeta,$$

where $n \in \mathbb{N}$. It will follow easily that

$$a := \int_0^z g_1(z)^n dz$$

is finite for all $n \in \mathbb{N}$, and using a result of W. K. Hayman [5, Lemma 1] we will see that $a = a(n) \neq 0$ if $n$ is sufficiently large. For such $n$ we then define $g_3(z) := g_2(z)/az$ and note that $g_3$ is of the form $g_3(z) = g_4(z^{q^2})$ for some entire function $g_4$. The function claimed in the theorem is

$$f(z) := zg_4(z)^{q-1}.$$

We remark that we introduced $z_0$ and $n$ only to ensure that $a \neq 0$. In a generic situation we could probably define $g_2$ directly via (2.7) with $z_0 = 0$ and $n = 1$.

To prove that $f$ has the desired properties, we determine the asymptotic behavior of the $g_j$ and $f$ in spiralling regions similar to $D$. We first note from (2.6) that if $0 < \eta_2 < \eta_1$ and if $0 < \theta_2 < \theta_1$, then

$$\log |g_0(re^{i \epsilon \log r + i \theta})| \leq -\eta_2 r^p \quad \text{for } |\theta| \leq \theta_2$$

and sufficiently large $r$. This implies that if $|\theta| \leq \theta_2/q$ and if $r$ is sufficiently large, then

$$\log |g_1(re^{i \epsilon \log r + i \theta})| = \log |g_0(r^q e^{i \epsilon \log (r^{q}) + i \theta})| \leq -\eta_2 r^{q^2}.$$
With
\[ S_1 := \left\{ re^{i\log r + i\theta} : r > 1, |\theta| < \frac{\theta_2}{q} \right\} \]
we thus find that
\[ |g_1(z)| \leq \exp(-\eta_2|z|^{q^p}) \]
if \( z \in S_1 \) is sufficiently large. Moreover, \( \sigma(t) \in S_1 \) for large \( t \), since for suitably chosen branches of the argument we have
\[ \arg \sigma(t) = \frac{1}{q} \arg(L(t_0 + t) - z_0) \]
\[ = \frac{1}{q} \arg L(t_0 + t) + o(1) \]
\[ = \frac{c}{q} \log |L(t_0 + t)| + o(1) \]
\[ = \frac{c}{q} \log |\sigma(t)| + o(1) \]
as \( t \to \infty \). From this it is straightforward to deduce that the integral defining \( a \) converges for all \( n \in \mathbb{N} \). In order to show that \( a \neq 0 \) for large \( n \), we use the following result of W. K. Hayman [5, Lemma 1].

**Lemma 3.** Let \( \gamma \) be a Jordan arc in \( \mathbb{C} \) which tends to \( \infty \) in both directions and let \( g \) be holomorphic in a domain containing \( \gamma \). Suppose that \( \int_{\gamma} |g(z)||dz| < \infty \) and that \( |g(z)| \to 0 \) as \( z \to \infty \) on \( \gamma \). Suppose also that \( |g(z)| \leq M \) for \( z \) on \( \gamma \), with equality for a single point \( z_1 \) on \( \gamma \) with \( g'(z_1) = 0 \). Suppose finally that \( \gamma \) cannot be deformed in a neighborhood of \( z_1 \) into a curve on which \( |g(z)| < M \). Then
\[ \int_{\gamma} g(z)^n dz \neq 0 \]
for all sufficiently large integers \( n \).

We remark that since \( g'(z_1) = 0 \), the set \( \{ z \in \mathbb{C} : |g(z)| < M \} \) has at least two components whose boundary contains \( z_1 \). The condition that \( \gamma \) cannot be deformed in a neighborhood of \( z_1 \) into a curve on which \( |g(z)| < M \) means that \( \gamma \) passes from one component of this set into another component at \( z_1 \).

We apply Lemma 3 with \( g := g_1 \), the curve \( \gamma \) parametrized as \( \gamma : \mathbb{R} \to \mathbb{C} \),
\[ \gamma(t) := \begin{cases} \sigma(-t) & \text{if } t \leq 0, \\ \sigma(t) := e^{2\pi i/q} \sigma(t) & \text{if } t > 0, \end{cases} \]
and \( z_1 := \gamma(0) = 0 \). Since by the choice of \( z_0 \) we have
\[ |g_1(\sigma^*(t))| = |g_1(\sigma(t))| = |\Pi(L(t_0 + t))| < |\Pi(L(t_0))| = |g_1(\sigma(0))| \]
for \( t > 0 \), it follows that \( |g_1(\gamma(t))| < |g_1(z_1)| \) for \( t \neq 0 \). Moreover, \( g_1'(z_1) = g_1'(0) = 0 \), and thus the hypotheses of Lemma 3 are satisfied. Since
\[ \int_{\gamma} g_1(z)^n dz = -\int_{\sigma} g_1(z)^n dz + \int_{\sigma^*} g_1(z)^n dz = (-1 + e^{2\pi i/q}) \int_{\sigma} g_1(z)^n dz \]
we conclude from Lemma 3 that \( a = \int_{\sigma} g_1(z)^n dz \neq 0 \) for sufficiently large values of \( n \).

Thus \( g_2(\sigma(t)) \to a \) as \( t \to \infty \). More generally, \( g_2(z) \to a \) as \( z \to \infty \) in \( S_1 \). In fact, if \( z \in S_1 \) then
\[ g_2(z) - a = \int_{\gamma} g_1(\zeta)^n d\zeta \]
for any path $\tau_z$ joining $z$ to $\infty$ in $S_1$. For large $z \in S_1$ and a suitable path $\tau_z$ we find that

$$|g_2(z) - a| \leq \left| \int_{\tau_z} g_1(\zeta)^n|d\zeta| \right| \leq \int_{\tau_z} \exp\left(-n\eta_2|\zeta|^q\right)|d\zeta| \leq \exp\left(-\eta_3|z|^q\right)$$

for some $\eta_3 > 0$. It follows that if $z \in S_1$ is sufficiently large, then

$$\left| g_3(z) - \frac{1}{z} \right| = \frac{|g_2(z) - a|}{|az|} \leq \frac{\exp\left(-\eta_3|z|^q\right)}{|az|} \leq \exp\left(-\eta_3|z|^q\right).$$

Now let

$$S_2 := \{ re^{ic\log r + i\theta} : r > 1, |\theta| < \theta_2 \}.$$

For $z \in S_2$ we have $\sqrt{\delta} \in S_1$ for a suitable branch. For large $z \in S_2$ we thus find that

$$|g_4(z) - \frac{1}{\sqrt{\delta}}| = \left| g_3(\sqrt{\delta}) - \frac{1}{\sqrt{\delta}} \right| \leq \exp\left(-\eta_3|z|^p\right);$$

i.e., if $z \in S_2$ is sufficiently large, then

$$f(z) - \sqrt{\delta} \leq \exp\left(-\eta_4|z|^p\right)$$

for some $\eta_4 > 0$ and a suitable branch.

3. **Newton’s Method for $f$**

We choose $\theta_3$ with $0 < \theta_3 < \theta_2$ and define

$$S_3 := \{ re^{ic\log r + i\theta} : r > 1, |\theta| < \theta_3 \}.$$

Then there exists $\delta > 0$ such that if $z \in S_3$ is sufficiently large, then the closed disk of radius $\delta|z|$ around $z$ is contained in $S_2$. With $d(z) := f(z) - \sqrt{\delta}$ we deduce from (2.8) that if $z \in S_3$ is sufficiently large, then

$$\left| \frac{f'(z) - \sqrt{\delta}}{q} \right| = \left| d'(z) \right|$$

$$\leq \frac{1}{\delta|z|} \left| \int_{[z = \delta|z|]} \frac{d(\zeta)}{(\zeta - z)^2} \right| \leq \frac{1}{\delta|z|} \left| \int_{[z = \delta|z|]} |d(\zeta)| \right| \leq \frac{1}{\delta|z|} \exp\left(-\eta_4(1 - \delta)^p|z|^p\right) \leq \exp\left(-\eta_5|z|^p\right)$$

for some $\eta_5 > 0$. Combining this with (2.8) we find that if $z \in S_3$ is sufficiently large, then

$$\left| \frac{f(z)}{f'(z)} - qz \right| \leq \exp\left(-\eta_6|z|^p\right)$$

where $\eta_6 > 0$. Since $q = p + 1$ we deduce that

$$|N(z) + pz| = \left| z - \frac{f(z)}{f'(z)} + pz \right| = \left| \frac{f(z)}{f'(z)} - qz \right| \leq \exp\left(-\eta_6|z|^p\right)$$

for large $z \in S_3$. In particular,

$$\left| |N(z)| - p|z| \right| \leq \exp\left(-\eta_6|z|^p\right)$$
which implies that
\[ (3.2) \quad | \log |N(z)| - \log(p|z|)| \leq \exp (-\eta_6|z|^\rho) \]
for large \( z \in S_3 \). Moreover, (3.1) yields
\[ (3.3) \quad |\arg N(z) - \arg(-pz)| \leq \exp (-\eta_6|z|^\rho) \]
for large \( z \in S_3 \). However, \( c \) was chosen such that \( c \log p = \pi \), so we deduce from (3.2) and (3.3) that
\[
| \arg N(z) - c \log |N(z)| | \\
\leq | \arg N(z) - \arg(-pz) | + | \arg(-pz) - c \log |p|z| | \\
+ |c \log |p|z| | - c \log |N(z)| | \\
\leq | \arg(-pz) - c \log |p|z| | + (1 + c) \exp (-\eta_6|z|^\rho) \\
\leq | \arg z + \pi - c \log p - c \log |z|| + (1 + c) \exp (-\eta_6|z|^\rho) \\
\leq | \arg z - c \log |z|| + (1 + c) \exp (-\eta_6|z|^\rho) \\
\leq | \arg z - c \log |z|| + \frac{1}{2|z|}
\]
for large \( z \in S_3 \). Since \( p > 23 \) we deduce from (3.1) that \( |N(z)| > 2|z| \) if \( z \in S_3 \) and if \( |z| \) is sufficiently large, say \( |z| > r_0 > 1 \). Combining this with the previous estimate, we conclude that if
\[
| \arg z - c \log |z|| < \theta_3 - \frac{1}{|z|},
\]
then
\[
| \arg N(z) - c \log |N(z)|| < \theta_3 - \frac{1}{|z|} + \frac{1}{2|z|} = \theta_3 - \frac{1}{2|z|} < \theta_3 - \frac{1}{|N(z)|}
\]
if \( z \in S_3 \) and if \( |z| \) is large enough, say \( |z| > r_1 > r_0 \). This implies that
\[
U := \left\{ re^{i \theta} : r > r_1, |\theta| < \theta_3 - \frac{1}{r} \right\}
\]
is \( N \)-invariant. Since \( |N(z)| \geq 2|z| \) for \( z \in U \), we have \( |N^k(z)| \geq 2^k|z| \) for \( z \in U \) and \( k \in \mathbb{N} \). Thus \( N^k|_U \to \infty \) as \( k \to \infty \). Hence \( U \) is contained in an invariant Baker domain of \( N \).

4. Asymptotic values of \( f \)

Suppose that \( f \) has a finite asymptotic value, say \( f(z) \to b \in \mathbb{C} \) as \( z \to \infty \) along a curve \( \Gamma \). The function
\[
F(z) := \frac{f(z)^q}{z}
\]
is entire since \( f(0) = 0 \). By (2.8) we have \( F(z) \to 1 \) as \( z \to \infty \) along the logarithmic spiral \( L \) while \( F(z) \to 0 \) as \( z \to \infty \) along \( \Gamma \). Thus \( F \) has two finite asymptotic values. By the Denjoy-Carleman-Ahlfors Theorem (see [7, §XI.4.5]), \( F \) has order at least 1. On the other hand, \( F \) has the same order as \( f \), which has been taken less than 1. This is a contradiction.
Remark. Our method will produce examples $f$ of any preassigned non-integer order $\rho \geq 1$, as well as examples with more than one invariant Baker domain. We only sketch the modifications that have to be made.

We again choose $\rho$ and $p$ such that (2.4) is holds. The condition $\mu < 1$ need not be satisfied, and there may be several, say $\ell$, intervals where $h(\theta)$ is negative and corresponding spiralling regions $S_1, \ldots, S_\ell$ where $\Pi(z) \to 0$ as $z \to \infty$. It is not difficult to see that $\ell$ can be any given positive number. For each $j$, let $L_j$ be a curve starting at 0 which outside the unit circle is a logarithmic spiral in $S_j$ and which inside the unit circle is a straight line from 0 to the corresponding point of the unit circle. Deforming one of the curves $L_j$ if necessary we may assume that there exists $z_0 \in \bigcup_{j=1}^\ell L_j$ such that $|\Pi(z_0)| > |\Pi(z)|$ for all $z \in \bigcup_{j=1}^\ell L_j$. Defining $g_2$ by (2.7) for some large $n$ and then $f$ as in §2, we find that $f(z) \sim c_j \sqrt[n]{z}$ for some $c_j \neq 0$ as $z \to \infty$ in $S_j$. As before, this means that $N(z) \sim -pz$ as $z \to \infty$ in $S_j$, $j = 1, \ldots, \ell$. We thus obtain an entire function $f$ for which $N$ has $\ell$ invariant Baker domains. A difference occurs in the proof that $f$ does not have finite asymptotic values. Here we cannot simply appeal to the classical Denjoy-Carleman-Ahlfors Theorem, but instead use that the function $f$ constructed has only $\ell$ "tracts"; see [6, §8.3].

Balasov’s result takes a different form if $\rho$ is an integer, but it seems possible to treat this case along the same lines.

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