PERMUTABLE ENTIRE FUNCTIONS SATISFYING ALGEBRAIC DIFFERENTIAL EQUATIONS

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Abstract. We show that if f and g are transcendental entire functions such that $f(g) = g(f)$, then f satisfies an algebraic differential equation if and only if g does.

1. Introduction

Let f and g be entire or rational functions. We say that f and g are *permutable* if $f(g) = g(f)$. Here we prove the following result.

Theorem. *Let f and g be permutable transcendental entire functions. If f satisfies an algebraic differential equation, then so does g.*

We describe some related results and background. First we note that permutability is obviously related to iteration since every function is clearly permutable with all its iterates. Fatou [6] and Julia [13] used the iteration theory they had developed to classify permutable polynomials (and in fact permutable rational functions whose Julia sets do not coincide with the Riemann sphere.) The permutable rational functions were then classified by Ritt [16]; a proof of Ritt's result using iteration theory was given by Eremenko [5]. The basic result obtained by these authors says that if two rational functions are permutable, then they have a common iterate, except in special cases arising from monomials, Chebychev polynomials and the multiplication theorems of certain elliptic functions. These exceptional cases can be described completely.

The classification of permutable transcendental entire functions is still an open problem. It was shown by Bargmann [2, §4] that if f and g have a common repelling fixed point, then the method of Fatou and Julia can be made to work: the Julia sets of f and g are equal, and if this common Julia set does not coincide with the whole plane, then f and g have a common iterate. Note, however, that in general two permutable entire functions need not have a common fixed point, as shown by the example $f(z) = z + e^z$ and $g(z) = z + e^z + 2\pi i$.

For many entire functions f the only nonlinear entire functions which are permutable with f are the iterates of f. Baker [1] showed that this is the case for $f(z) = ae^{bz} + c$, where $a, b, c \in \mathbb{C}$, $a, b \neq 0$. Ng [14] gave a fairly general class of entire functions f for which any nonlinear entire function g permutable with f is of the form $g(z) = af^n(z) + b$, where a is a root of unity, $b \in \mathbb{C}$, $n \in \mathbb{N} := \{1, 2, 3, \dots\}$, and f^n is the n-th iterate of f. Ng's class contains all functions of the form $f(z) = e^z + p(z)$ or $f(z) = \sin z + p(z)$ where p is a nonconstant polynomial, but it does not contain Baker's example $f(z) = ae^{bz} + c$.

2000 *Mathematics Subject Classification.* Primary 30D05; Secondary 34M05, 39B12.

Key words and phrases. Permutable, commuting, iteration, factorization, differential equation.

Supported by the Alexander von Humboldt Foundation and by the G.I.F., the German–Israeli Foundation for Scientific Research and Development, Grant G -809-234.6/2003.

There are several papers in which for certain particular functions f the set of all entire functions g of finite order which are permutable with f is determined; see [12, 18, 19, 20, 21, 22, 23]. In some of these papers, notably [23], the fact that the given function f satisfies some differential equation is used. In particular, it is proved in [23, Theorem 1] that if f and g are permutable entire functions of finite order, and if f satisfies a linear differential equation with rational coefficients, then g satisfies such a differential equation. Our theorem above can be considered as a further contribution in this direction, with the restriction on the order of g and on the linearity of the differential equation removed.

The main tools used are a result of Baker [1] relating the growth of permutable entire functions (Lemma 1) and a result of Steinmetz [17] concerning factorization of solutions of certain functional and differential equations (Lemma 3). Steinmetz's result was also used in [23]. We shall use the standard terminology of the theory of entire and meromorphic functions as given in [11].

2. Lemmas

The following result due to Baker [1, Satz 7] is essential for our argument.

Lemma 1. *Let f and g be permutable transcendental entire functions. Then there exists $p \in \mathbb{N}$ and $R > 0$ such that $M(r, g) < M(r, f^p)$ for $r > R$.*

A consequence is the following result.

Lemma 2. *Let f and g be permutable transcendental entire functions. Then there exists $q \in \mathbb{N}$ and $R > 0$ such that $T(r, g) < T(r, f^q)$ for $r > R$.*

Proof. By a classical result of Pólya [15] (see also [4, Theorem 6] and [11, Theorem 2.9]) there exists a constant $c > 0$ such that if F and G are entire, then

$$M\left(r, F\left(G\right)\right) \geq M\left(cM\left(\frac{r}{2}, G\right), F\right)$$

for sufficiently large r. Moreover, for entire F we have

$$T(r, F) \leq \log M(r, F) \leq 3T(2r, F)$$

for sufficiently large r; see [11, Theorem 1.6]. Combining these estimates we obtain with p as in Lemma 1

$$\begin{aligned} T\left(r, f^{p+1}\right) &\geq \frac{1}{3}\log M\left(\frac{r}{2}, f^{p+1}\right) \\ &\geq \frac{1}{3}\log M\left(cM\left(\frac{r}{4}, f\right), f^{p}\right) \\ &\geq \frac{1}{3}\log M\left(cM\left(\frac{r}{4}, f\right), g\right) \\ &\geq \frac{1}{3}T\left(cM\left(\frac{r}{4}, f\right), g\right) \end{aligned}$$

for large r. Now $cM(\frac{r}{4}, f) > r^4$ for sufficiently large r since f is transcendental, and $T(r^4, g) \geq 3T(r, g)$ for large r since $T(r, g)$ is convex in $\log r$. The conclusion follows with $q := p + 1$. □

Another important tool we shall use is the following result of Steinmetz [17, Satz 1, Korollar 1]. Generalizations and different proofs of this result have been given by Brownawell [3] and Gross and Osgood [7, 8, 9].

Lemma 3. *Let F_0,F_1,...,F_n be not identically vanishing meromorphic functions and let h_0,h_1,...,h_n be meromorphic functions that do not all vanish identically. Let g be a nonconstant entire function and suppose that there exists a positive constant K such that $\sum_{j=0}^{n} T(r,h_j) \leq KT(r,g)$ as $r \to \infty$ outside some exceptional set of finite measure. Suppose also that $\sum_{j=0}^{n} h_j F_j(g) = 0$. Then there exist polynomials p_0,p_1,...,p_n that do not all vanish identically such that $\sum_{j=0}^{n} p_j F_j = 0$.*

We also require some results about differential polynomials. Let $n \in \mathbb{N}$, $m_j \in \mathbb{N}_0 := \mathbb{N} \cup \{0\}$ for $j = 0, 1, \ldots, n$, and put $m = (m_0, m_1, \ldots, m_n)$. Define $M_m[f]$ by

$$M_m[f](z) = f(z)^{m_0} f'(z)^{m_1} f''(z)^{m_2} \ldots f^{(n)}(z)^{m_n},$$

with the convention that $M_{(0)}[f] = 1$. We call $w(m) = m_1 + 2m_2 + \cdots + nm_n$ the *weight* of $M_m[f]$. A *differential polynomial* $P[f]$ is an expression of the form

$$P[f](z) = \sum_{m \in I} a_m(z) M_m[f](z),$$

where the a_m are meromorphic functions called the *coefficients* of $P[f]$ and I is a finite index set. The *weight* $w(P)$ of $P[f]$ is given by $w(P) = \max_m w(m)$, where the maximum is taken over all $m \in I$ for which $a_m \not\equiv 0$.

Of course, a meromorphic function f is said to satisfy an algebraic differential equation if there exists a nontrivial differential polynomial $P[f]$ with rational coefficients such that $P[f] = 0$.

Lemma 4. *Let f and g be permutable transcendental entire functions. Let $P[f] = \sum_{m \in I} a_m M_m[f]$ be a differential polynomial with rational coefficients a_m. Then $P[f](g) = \sum_{m \in I} a_m(g) M_m[f](g) = 0$ can be written in the form*

$$P[f](g) = \sum_{m \in J} b_m M_m[g](f),$$

where the b_m are meromorphic functions satisfying $T(r, b_m) = O(U(r))$ as $r \to \infty$ outside some exceptional set of finite measure, with

$$U(r) = \max\{T(r,f), T(r,g)\}.$$

Moreover, $\max_{m \in J} w(m) = w(P)$.

Proof. Differentiating $f(g) = g(f)$ we find that

$$f'(g) = g'(f) \frac{f'}{g'},$$

$$f''(g) = g''(f) \frac{(f')^2}{(g')^2} + g'(f) \frac{f''}{(g')^2} - g'(f) \frac{f'g''}{(g')^3},$$

and so forth. Substituting this in $P[f](g)$ yields the desired representation of $P[f](g)$ with coefficients b_m which are rational functions of g and of the derivatives of f and g up to order n. Since $T(r, f^{(k)}) \leq (1 + o(1))T(r,f)$ for each $k \in \mathbb{N}$ as $r \to \infty$ outside some exceptional set of finite measure (see [11, p. 56]) the conclusion follows. □

Finally we recall the following result of Ostrowski [10, p. 269].

Lemma 5. *Let f and g be analytic in certain domains, with f defined in the range of g. Suppose that f and g both satisfy some (possibly different) algebraic differential equation. Then $f(g)$ satisfies some algebraic differential equation.*

In general, of course, the differential equation for $f(g)$ will be "more complicated" than those for f and g; that is, its weight will be larger.

3. Proof of the theorem

Let f and g be permutable transcendental entire functions and suppose that f satisfies an algebraic differential equation.

We choose q according to Lemma 2 and put $F := f^q$. Lemma 5 implies that F also satisfies an algebraic differential equation, say

$$P[F] = \sum_{m \in I} c_m M_m[F] = 0,$$

with rational functions c_m. By Lemma 4 we have

$$0 = P[F](g) = \sum_{m \in J} b_m M_m[g](F)$$

for suitable meromorphic functions b_m satisfying $T(r, b_m) = O(U(r))$ as $r \to \infty$ outside some exceptional set of finite measure, where

$$U(r) = \max \{T(r, F), T(r, g)\}.$$

By Lemma 2 and our choice of q we have $T(r, g) < T(r, F)$ for large r. We deduce that $T(r, b_m) = O(T(r, F))$ as $r \to \infty$ outside some exceptional set of finite measure. It now follows from Lemma 3 that there exists polynomials p_m such that

$$\sum_{m \in J} p_m M_m[g] = 0.$$

Thus g satisfies an algebraic differential equation. □

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