FIXED POINTS OF COMPOSITE ENTIRE AND QUASIREGULAR MAPS

WALTER BERGWEILER

Abstract. We give a new proof of the result that if f and g are entire transcendental functions, then $f \circ g$ has infinitely many fixed points. The method yields a number of generalizations of this result. In particular, it extends to quasiregular maps in $\mathbb{R}^d$.

1. Introduction and main results

The following result was conjectured by Gross (see [11, p. 542] and [15, Problem 5]) and first proved in [4].

Theorem A. *Let f and g be entire transcendental functions. Then $f \circ g$ has infinitely many fixed points.*

The following generalization of Theorem A was proved in [5].

Theorem B. *Let f and g be entire transcendental functions. Then $f \circ g$ has infinitely many repelling fixed points.*

Here a fixed point ξ of a holomorphic function h is called repelling if $|h'(\xi)| > 1$. The repelling fixed points play an important role in iteration theory.

The purpose of this paper is twofold. Firstly, we give a new proof of Theorem A. Secondly, we obtain some generalizations of Theorem A (and B).

The main difference between the method employed here and the previous proofs of Theorem A and B is that the Wiman-Valiron method which was crucial in [4, 5] is not used here. Instead we use some ideas from normal families. This method is also applicable for quasiregular maps; see [23] for the definition and basic properties of quasiregular maps.

Theorem 1. *Let $d \geq 2$ and let $f, g : \mathbb{R}^d \to \mathbb{R}^d$ be quasiregular maps with an essential singularity at ∞. Then $f \circ g$ has infinitely many fixed points.*

For functions in the plane we also obtain some extensions of the previously known results.

Theorem 2. *Let f and g be entire transcendental functions. Then there exists a sequence (ξ_n) such that $(f \circ g)(\xi_n) = \xi_n$ and $(f \circ g)'(\xi_n) \to \infty$.*

Theorem 3. *Let f and g be entire transcendental functions. Then $f \circ g$ has infinitely many nonreal fixed points.*

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Theorem 3 answers a question of Clunie [10] who had shown that at least one of the two functions $f \circ g$ and $g \circ f$ has infinitely many nonreal fixed points. The special case $f = g$ had been dealt with earlier in [8], answering a question of Baker [2]. Theorem 3 implies that for any straight line there are infinitely many fixed points not lying on this line.

Similar ideas to the ones employed in this paper were used – in the context of iteration rather than composition – in [3, 7, 12, 13] for holomorphic maps, and in [25, 26] for quasiregular maps.

Although each of the Theorems 1–3 contains Theorem A as a special case, we will first give a proof of Theorem A in §2, as this explains the underlying idea best. In §§3–5 we will then prove Theorems 1–3. These sections will make occasional reference to §2, but are independent of each other.

2. Proof of Theorem A

2.1. **Preliminary Lemmas.** We shall need a result from the Ahlfors theory of covering surfaces; see [1], [17, Chapter 5] or [21, Chapter XIII] for an account of this theory. To state the result of the Ahlfors theory that we need, let $D \subset \mathbb{C}$ be a domain and let $f : D \to \mathbb{C}$ be holomorphic. Given a Jordan domain $V \subset \mathbb{C}$, we say that f has an *island* over V if $f^{-1}(V)$ has a component whose closure is contained in D. Note that if U is such a component, then $f|_U : U \to V$ is a proper map.

Lemma 2.1. *Let $D \subset \mathbb{C}$ be a domain and let $D_1, D_2 \subset \mathbb{C}$ be Jordan domains with disjoint closures. Let $\mathcal{F}$ be a family of functions holomorphic in D which is not normal. Then there exists a function $f \in \mathcal{F}$ which has an island over D_1 or D_2.*

For example, Lemma 2.1 follows from Theorem 5.5 (applied with a domain D_3 containing ∞) and Theorem 6.6 in [17].

For a different proof of Lemma 2.1 see [6, §5.1]. The proof given there is particularly simple in the case where the D_j are small disks. It turns out that this special case suffices for our purposes.

The following lemma is a simple consequence of the maximum principle.

Lemma 2.2. *Let $D \subset \mathbb{C}$ be a domain and let (f_n) be sequence of functions holomorphic in D which is not normal. If (f_n) converges locally uniformly in $D \backslash E$ for some finite set E, then $f_n \to \infty$ in $D \backslash E$.*

This lemma will be useful when dealing with quasinormal families. By definition, a family $\mathcal{F}$ of functions holomorphic in a domain D is called *quasinormal* (cf. [9, 20, 24]) if for each sequence (f_n) in $\mathcal{F}$ there exists a subsequence (f_{n_k}) and a finite set $E \subset D$ such that (f_{n_k}) converges locally uniformly in $D \backslash E$. If the cardinality of the exceptional set E can be bounded independently of the sequence (f_n), and if q is the smallest such bound, then we say that $\mathcal{F}$ is quasinormal of *order* q.

We denote the maximum modulus of an entire function f by $M(r, f)$.

Lemma 2.3. *Let f be an entire transcendental function and $A > 1$. Then*

$$\lim_{r \to \infty} \frac{M(Ar, f)}{M(r, f)} = \infty.$$

This result follows easily from the convexity of $\log M(r, f)$ in $\log r$ and the transcendency of f. We omit the details. For an alternative proof of Lemma 2.3 see the proof of Lemma 3.3 in §3.1 below.

2.2. **Proof of Theorem A.** We first choose a sequence (c_n) tending to ∞ such that $|f(c_n)| \leq 1$. We may assume that $|c_n| \geq |g(0)|$ for all n and define r_n by $M(r_n, g) = |c_n|$.

We define

$$f_n(z) := \frac{f(c_n z)}{r_n} \quad \text{and} \quad g_n(z) := \frac{g(r_n z)}{c_n}.$$

It is easy to see that no subsequence of (f_n) is normal at 0. Since $f_n(1) \to 0$ it follows from Lemma 2.2 that (f_n) is not normal in $\mathbb{C}\backslash\{0\}$. Passing to a subsequence if necessary we may thus assume that no subsequence of (f_n) is normal at $a_1 := 0$ and some $a_2 \in \mathbb{C}\backslash\{0\}$.

It follows from Lemma 2.3 that if $n \to \infty$, then $M(r, g_n) \to 0$ if $r < 1$ and $M(r, g_n) \to \infty$ if $r > 1$. Lemma 2.2 implies that the sequence (g_n) is not quasinormal. Passing to a subsequence if necessary we may thus assume that there exist $b_1, b_2, b_3 \in \mathbb{C}\backslash\{0\}$ where no subsequence of (g_n) is normal. We choose $0 < \varepsilon < \frac{1}{2}$ such that the closed disks of radius ε around the b_j are pairwise disjoint and do not contain 0. In the following we denote by $B(a, r)$ the open disk of radius r around a point a; that is, $B(a, r) := \{z \in \mathbb{C} : |z - a| < r\}$.

It follows from Lemma 2.1 that if n is sufficiently large and $j \in \{1, 2\}$, then f_n has an island in $B(a_j, \varepsilon)$ over at least two of the domains $B(b_k, \varepsilon)$. This implies that there exists $k \in \{1, 2, 3\}$ such that f_n has an island $U_1 \subset B(a_1, \varepsilon)$ and another island $U_2 \subset B(a_2, \varepsilon)$ over the same disk $B(b_k, \varepsilon)$.

Moreover, it follows from Lemma 2.1 that if n is sufficiently large, then there exists $j \in \{1, 2\}$ such that g_n has an island $V \subset B(b_k, \varepsilon)$ over $B(a_j, \varepsilon)$. Then $V \cap g_n^{-1}(U_j)$ contains a component W of $(f_n \circ g_n)^{-1}(B(b_k, \varepsilon))$ satisfying $\overline{W} \subset B(b_k, \varepsilon)$.

For $z \in \partial W$ we have

$$|((f_n \circ g_n)(z) - b_k) - ((f_n \circ g_n)(z) - z)| = |z - b_k| < \varepsilon = |(f_n \circ g_n)(z) - b_k|.$$

Rouché's theorem implies that the number of fixed points of $f_n \circ g_n$ in W coincides with the number of zeros of $f_n \circ g_n - b_k$ in W. As $f_n \circ g_n$ is a proper map from W onto $B(b_k, \varepsilon)$ the function $f_n \circ g_n - b_k$ has at least one zero in W and thus $f_n \circ g_n$ has a fixed point $\xi \in W \subset B(b_k, \varepsilon)$. Then $\xi_n := r_n \xi$ is a fixed point of $f \circ g$. Since $\xi_n \in B(r_n b_k, r_n \varepsilon)$ it follows that $\xi_n \to \infty$ as $n \to \infty$ so that $f \circ g$ has infinitely many fixed points. $\square$

3. Proof of Theorem 1

3.1. **Preliminary Lemmas.** As a general reference for quasiregular maps we recommend [23]. We first state some lemmas analogous to those stated in §2.1, and begin with the analogue of Lemma 2.1.

We note that Lemma 2.1 is a generalization of Montel's theorem, which in turn is the result that corresponds to Picard's theorem in the context of normal families. The analogue of Picard's theorem for quasiregular maps was given by Rickman [22] who proved that there exists $q = q(d, K) \in \mathbb{N}$ with the property that every K-quasiregular map $f : \mathbb{R}^d \to \mathbb{R}^d$ which omits q points is constant. We shall call this number q the *Rickman constant*. The corresponding normality result was proved by Miniowitz [19], using an extension of the Zalcman lemma [27] to quasiregular maps. We refer to [19] also for further information about normal families of quasiregular maps. Besides normal families we will also consider quasinormal families of quasiregular maps, which are defined in exactly the same way as for holomorphic functions.

Miniowitz's extension of the Zalcman lemma has been used by Siebert [25, 26] to deduce the following Lemma 3.1 from Rickman's theorem. Here we call, as in §2.1, a domain U an *island* of the quasiregular map $f : D \to \mathbb{R}^d$ over the simply connected domain $V \subset \mathbb{R}^d$, if U is a component of $f^{-1}(V)$ and if $\overline{U} \subset D$. And as in dimension 2 we denote by $B(a,r)$ the open ball of radius r around a point $a \in \mathbb{R}^d$; that is, $B(a,r) := \{x \in \mathbb{R}^d : |x-a| < r\}$. Here $|x|$ is the (Euclidean) norm of a point $x \in \mathbb{R}^d$. With this notation Siebert's result (see [25, Satz 2.2.2] or [26, Corollary 3.2.2]) can be stated as follows.

Lemma 3.1. *Let $d \geq 2$, $K \geq 1$ and let $q = q(d,K)$ be the Rickman constant. Let $a_1, \dots, a_q \in \mathbb{R}^d$ be distinct. Then there exists $\varepsilon > 0$ with the following property: if $D \subset \mathbb{R}^d$ is a domain and $\mathcal{F}$ is a non-normal family of functions K-quasiregular in D, then there exists a function $f \in \mathcal{F}$ which has an island over $B(a_j, \varepsilon)$ for some $j \in \{1, \dots, q\}$.*

The following lemmas is literally the same as Lemma 2.2 in §2.1, and it is again a simple consequence of the maximum principle.

Lemma 3.2. *Let $D \subset \mathbb{R}^d$ be a domain and let (f_n) be non-normal sequence of functions which are K-quasiregular in D. If (f_n) converges locally uniformly in $D \backslash E$ for some finite set E, then $f_n \to \infty$ in $D \backslash E$.*

The next lemma is identical to Lemma 2.3 in §2.1. Again we denote by $M(r,f)$ the maximum modulus; that is, $M(r,f) := \max_{|x|=r} |f(x)|$.

Lemma 3.3. *Let $f : \mathbb{R}^d \to \mathbb{R}^d$ be quasiregular with an essential singularity at ∞ and let $A > 1$. Then*

$$\lim_{r \to \infty} \frac{M(Ar,f)}{M(r,f)} = \infty.$$

Proof. Suppose that the conclusion does not hold. Then there exist $C > 1$ and a sequence (r_n) tending to ∞ such that $M(Ar_n, f) \leq CM(r_n, f)$. The sequence (f_n) defined by

$$f_n(x) := \frac{f(r_n x)}{M(r_n, f)}$$

is then bounded and thus normal in $B(0,A)$. Passing to a subsequence we may assume that $f_n \to h$ for some quasiregular map $h : B(0,A) \to \mathbb{R}^d$. We have $h(0) = 0$ while $M(1,h) = 1$. Thus h is not constant.

Now there exists $a \in \mathbb{R}^d$ such that f has infinitely many a-points. Without loss of generality we may assume that $a = 0$ since otherwise we can consider $f(x+a) - a$ instead of $f(x)$. A contradiction will now be obtained from Hurwitz's theorem (cf. [19, Lemma 2]).

More precisely, choose $0 < t < 1$ such that $h(x) \neq 0$ for $|x| = t$. For sufficiently large n we then have $\mu(h, B(0,t), 0) = \mu(f_n, B(0,t), 0) = \mu(f, B(0, r_n t), 0)$. Here $\mu(h, B(0,t), 0)$ denotes the topological degree. Thus

$$\mu(h, B(0,t), 0) = \sum_{x \in h^{-1}(0) \cap B(0,t)} i(x,h)$$

where $i(x,h)$ is the topological index. But $\mu(f, B(0, r_n t), 0) \to \infty$ as $n \to \infty$ since f has infinitely many zeros. This is a contradiction. $\square$

The next lemma is a simple consequence of Lemma 3.3.

Lemma 3.4. *Let $f : \mathbb{R}^d \to \mathbb{R}^d$ be quasiregular with an essential singularity at ∞. Then*
$$\lim_{r\to\infty} \frac{\log M(r,f)}{\log r} = \infty.$$

The following lemma (see [25, Lemma 1.3.14] or [26, Lemma 2.1.5]) replaces the argument where Rouché's theorem was used in the proof of Theorem A.

Lemma 3.5. *Let $U \subset \mathbb{R}^d$ be a domain and let $a \in \mathbb{R}^d$ and $r > 0$ be such that $\overline{U} \subset B(a,r)$. Suppose that $h : U \to B(a,r)$ is proper and quasiregular. Then h has a fixed point in U.*

Next we shall need the following result.

Lemma 3.6. *Let $K > 1$, let $D \subset \mathbb{R}^d$ a domain and let C be a compact subset of D. Then there exist $\alpha, \beta > 0$ with the following property: if f is K-quasiregular in D and satisfies $|f(x)| \geq 1$ for all $x \in D$, then $\log|f(y)| \leq \alpha + \beta \log|f(x)|$ for all $x, y \in C$.*

Proof. It follows from [23, Corollary 3.9, p. 91] that there exist $A, B > 0$ such that if $B(a, 2\delta) \subset D$, then $\log|f(x)| \leq A + B\log|f(a)|$ for all $x \in B(a,\delta)$. We may assume that C is connected. Since C is compact there exist $a_1, \dots, a_N \in C$ and $\delta > 0$ with
$$C \subset \bigcup_{j=1}^{N} B(a_j, \frac{1}{2}\delta) \quad \text{and} \quad \bigcup_{j=1}^{N} B(a_j, 2\delta) \subset D.$$
The conclusion follows with $\beta := B^{N+1}$ and some α. □

Finally we need the following observation apparently made first in [14, Lemma 3].

Lemma 3.7. *Let A, B be sets and let $f : A \to B$ and $g : B \to A$ be functions. Then the set of fixed points of $f \circ g$ and the set of fixed points of $g \circ f$ have the same cardinality.*

To prove this lemma we only have to observe that g is a bijection from the set of fixed points of $f \circ g$ to the set of fixed points of $g \circ f$.

3.2. **Proof of Theorem 1.** Let (c_n) be a sequence in $\mathbb{R}^d$ which tends to ∞ and define $F_n(x) := f(|c_n|x)/|c_n|$ and $G_n(x) := g(|c_n|x)/|c_n|$. Lemma 3.4 yields that $M(r, F_n) \to \infty$ and $M(r, G_n) \to \infty$ as $n \to \infty$ if $r > 0$, while $F_n(0) \to 0$ and $G_n(0) \to 0$. Thus no subsequence of (F_n) or (G_n) is normal at 0.

We distinguish between two cases.

Case 1. For every choice of (c_n) the sequences (F_n) and (G_n) are both quasinormal.

We may choose the sequence (c_n) such that $|g(c_n)| \leq 1$ for all n. Applying Lemma 3.2 and, passing to subsequences if necessary, we may assume that $F_n \to \infty$ in $\mathbb{R}^d \backslash E_f$ and that $G_n \to \infty$ in $\mathbb{R}^d \backslash E_g$ for two finite sets E_f and E_g containing 0. Moreover, we may assume that E_g contains at least one point $b \in \mathbb{R}^d$ with $|b| = 1$, with no subsequence of (G_n) converging in a neighborhood of b.

We choose $\varepsilon > 0$ such that $2\varepsilon < |a - b|$ for all $a \in E_g \backslash \{b\}$ and $2\varepsilon < |a|$ for all $a \in E_f \backslash \{0\}$. For sufficiently large n we then have $|G_n(x)| > 2$ for $x \in \partial B(b,\varepsilon)$ and $|F_n(x)| > 2$ for $x \in \partial B(0,\varepsilon)$, while $|G_n(y)| < 1$ for some $y \in B(b,\varepsilon)$ and $|F_n(z)| < 1$ for some $z \in B(0,\varepsilon)$. Since $B(0,\varepsilon) \cup B(b,\varepsilon) \subset B(0,2)$ this implies that G_n has an island $V \subset B(b,\varepsilon)$ over $B(0,\varepsilon)$ while F_n has an island $U \subset B(0,\varepsilon)$ over $B(b,\varepsilon)$. As in the proof of Theorem A we find that $V \cap G_n^{-1}(U)$ contains a component W of $(F_n \circ G_n)^{-1}(B(b,\varepsilon))$ satisfying $\overline{W} \subset B(b,\varepsilon)$. Lemma 3.5 now implies that $F_n \circ G_n$ has a fixed point in $\xi \in B(b,\varepsilon)$ and thus $f \circ g$ has a fixed point $\xi_n \in B(|c_n|b, |c_n|\varepsilon)$. It follows that $f \circ g$ has infinitely many fixed points.

Case 2. The sequence (c_n) can be chosen such that one of the sequences (F_n) and (G_n) is not quasinormal.

Because of Lemma 3.7 we may assume that the sequence (F_n) is not quasinormal. Passing to a subsequence if necessary, we may in fact assume that no subsequence of (F_n) is quasinormal.

As in the proof of Theorem A we may assume that $|c_n| \geq |g(0)|$ and define r_n by $M(r_n, g) = |c_n|$. As there we also define $f_n(x) := f(|c_n|x)/r_n = F_n(x)|c_n|/r_n$ and $g_n(x) := g(r_n x)/|c_n|$. Again we find that no subsequence of (f_n) is normal at 0.

We now show that (f_n) is not quasinormal. To do this we assume that (f_n) is quasinormal. Passing to a subsequence we then may assume that $f_n \to \infty$ in $\mathbb{R}^d \backslash E$ for some finite set E. Let $C \subset \mathbb{R}^d \backslash E$ be a compact set containing $\partial B(0,r)$ for some $r > 0$. Then there exists a domain $D \supset C$ such that $|f_n(x)| \geq 1$ for $x \in D$ if n is large. Lemma 3.6 yields that

$$\log M(r, f_n) \leq \alpha + \beta \log |f_n(x)| \tag{3.1}$$

for $x \in C$ and large n. On the other hand, Lemma 3.4 implies that

$$\log M(r, f_n) = \log M(|c_n|r, f) - \log r_n \geq 4\beta \log(|c_n|r) - \log r_n$$

if n is large. Lemma 3.4 also yields $\beta \log |c_n| = \beta \log M(r_n, g) \geq \log r_n$ for large n. Thus $\log M(r, f_n) \geq 3\beta \log |c_n| + 4\beta \log r \geq 2\beta \log |c_n|$ for large n. We deduce from this and (3.1) that

$$\log |f_n(x)| \geq \frac{\log M(r, f_n) - \alpha}{\beta} \geq 2 \log |c_n| - \frac{\alpha}{\beta} \geq \log |c_n|$$

for $x \in C$ and large n. It follows that

$$\log |F_n(x)| = \log |f_n(x)| - \log |c_n| + \log r_n \geq \log r_n$$

for $x \in C$ and large n. Hence $F_n \to \infty$ in $\mathbb{R}^d \backslash E$, contradicting the assumption that no subsequence of (F_n) is quasinormal. Thus (f_n) is not quasinormal. Passing to a subsequence if necessary we may assume that no subsequence of (f_n) is quasinormal.

From Lemma 3.3 we deduce that $M(r, g_n) \to 0$ if $r < 1$ and $M(r, g_n) \to \infty$ if $r > 1$. Lemma 3.2 now implies that no subsequence of (g_n) is quasinormal.

Let K be such that f and g are K-quasiregular and define $p := 2q - 1 \geq q$ where $q = q(d, K)$ is the Rickman constant. Passing to subsequence if necessary we may assume that there exist $a_1, \ldots, a_p \in \mathbb{R}^d \backslash \{0\}$ where no subsequence of (f_n) is normal and that there exist $b_1, \ldots, b_p \in \mathbb{R}^d \backslash \{0\}$ where no subsequence of (g_n) is normal.

It follows from Lemma 3.1 that there exists $\varepsilon > 0$ such that if n is sufficiently large and $j \in \{1, \ldots, p\}$, then f_n has an island in $B(a_j, \varepsilon)$ over at least $p - q + 1$ of the p balls $B(b_k, \varepsilon)$, and g_n has an island in $B(b_j, \varepsilon)$ over at least $p - q + 1$ of the p balls $B(a_k, \varepsilon)$.

This implies that there exists $k \in \{1, \ldots, p\}$ such that f_n has an island in $B(a_j, \varepsilon)$ over $B(b_k, \varepsilon)$ for at least $p - q + 1 = q$ values of j. Lemma 3.1 implies that for at least one such value j the function g_n has an island in $B(b_k, \varepsilon)$ over $B(a_j, \varepsilon)$.

Thus we obtain $j, k \in \{1, \ldots, p\}$ such that f_n has an island $U \subset B(a_j, \varepsilon)$ over $B(b_k, \varepsilon)$ and g_n has an island $V \subset B(b_k, \varepsilon)$ over $B(a_j, \varepsilon)$. As before we find that $V \cap g_n^{-1}(U)$ contains a component W of $(f_n \circ g_n)^{-1}(B(b_k, \varepsilon))$ satisfying $\overline{W} \subset B(b_k, \varepsilon)$.

Lemma 3.5 now implies that $f_n \circ g_n$ has a fixed point in $B(b_k, \varepsilon)$, and thus $f \circ g$ has a fixed point in $B(|c_n| b_k, |c_n| \varepsilon)$. Thus $f \circ g$ has infinitely many fixed points. □

4. Proof of Theorem 2

4.1. **Preliminary Lemmas.** We shall require an additional result from the Ahlfors theory. An island U of a function f over a domain V is called *simple* if $f : U \to V$ is univalent. The following result can also be found in the references given in §2.1.

Lemma 4.1. *Let $D \subset \mathbb{C}$ be a domain and let D_1, D_2 and D_3 be Jordan domains with pairwise disjoint closures. Let $\mathcal{F}$ be a non-normal family of functions holomorphic in D. Then there exists a function $f \in \mathcal{F}$ which has a simple island over D_1, D_2 or D_3.*

It follows from Lemma 4.1 that a non-constant entire function f has a simple island over one of three Jordan domains D_1, D_2, D_3 with pairwise disjoint closures. We shall need the simple observation made in [7] that we need only two domains D_1, D_2 if f is a polynomial or, more generally, a proper holomorphic map whose range contains D_1 and D_2.

Although the formulation of the following Lemma 4.2 was slightly different in [7, Lemma 2.2], we omit the simple proof based on the Riemann-Hurwitz formula, but note that it is analogous to the proof of Lemma 4.3 which we will give below.

Lemma 4.2. *Let $f : U \to V$ be a proper holomorphic map and let D_1 and D_2 be Jordan domains with disjoint closures contained in V. Then there exist two domains $U_1, U_2 \subset U$ which are simple islands over D_1 or D_2.*

Here U_1 and U_2 need not be islands over the same domain. We allow the possibility that U_1 is an island over D_1 and U_2 is an island over D_2, or vice versa. For example, this will always be the case if f is univalent. For proper maps of higher degree, however, we have the following lemma.

Lemma 4.3. *Let $f : U \to V$ be a proper holomorphic map of degree at least 2 and let D_1, D_2 and D_3 be Jordan domains with pairwise disjoint closures contained in V. Then there exists $k \in \{1, 2, 3\}$ such that f has two simple islands over D_k.*

Proof. Let $U_1, \dots, U_m$ be the components of $f^{-1}\left(\bigcup_{k=1}^3 D_k\right)$. Thus the U_j are the islands over the domains D_k. Now $f|_{U_j}$ is a proper map of some degree μ_j and $\sum_{j=1}^m \mu_j = 3d$, where d is the degree of f. By the Riemann-Hurwitz formula the number of critical points contained in U_j is $\mu_j - 1$, and f has $d-1$ critical points in U. Thus

$$3d - m = \sum_{j=1}^{m} (\mu_j - 1) \le d - 1$$

so that $m \ge 2d + 1$. Since f has $d - 1$ critical points in U we conclude that the number n of domains U_j which do not contain a critical point satisfies

$$n \ge m - (d-1) \ge (2d+1) - (d-1) = d + 2 \ge 4.$$

Thus among the U_j there at least 4 simple islands, and hence two of them must be over the same domain D_k. □

We shall also use the following well-known result; see, e.g., [7, Lemma 2.3] for the simple proof.

Lemma 4.4. *Let $0 < \delta < \frac{\varepsilon}{2}$ and let $U \subset B(a, \delta)$ be a simply-connected domain. Let $f : U \to B(a, \varepsilon)$ be holomorphic and bijective. Then f has a fixed point ξ in U which satisfies $|f'(\xi)| \geq \varepsilon/4\delta$.*

We shall also need the following lemma concerning entire functions of small growth.

Lemma 4.5. *Let g be an entire function of the form*

$$g(z) = \prod_{k=1}^{\infty} \left(1 - \frac{z}{z_k}\right)$$

where $0 < |z_1| \leq |z_2| \leq \ldots$ and $\lim_{k\to\infty} |z_{k+1}/z_k| = \infty$. Denote the zeros of g' by z'_k, ordered such that $0 \leq |z'_1| \leq |z'_2| \leq \ldots$. Then $\lim_{k\to\infty} |z'_{k+1}/z'_k| = \infty$.

Proof. For sufficiently large n there exists r satisfying $|z_n| \leq r$ and $8r \leq |z_{n+1}|$. We show first that for such r

$$\min_{|z|=4r} |g(z)| > \max_{|z|=r} |g(z)|, \tag{4.1}$$

provided n is large enough. Let $|u| = r$ and $|v| = 4r$. We will show that $|g(v)| > |g(u)|$. To this end we write

$$\begin{aligned} \log \frac{|g(v)|}{|g(u)|} &= \sum_{k=1}^{\infty} \log \frac{|v - z_k|}{|u - z_k|} \\ &= \sum_{k=1}^{n} \log \frac{|v - z_k|}{|u - z_k|} + \sum_{k=n+1}^{\infty} \log \frac{|v - z_k|}{|u - z_k|} \\ &= S_1 + S_2. \end{aligned}$$

For $k \leq n$ we have $|z_k| \leq r < |v|$ so that

$$\frac{|v - z_k|}{|u - z_k|} \geq \frac{|v| - |z_k|}{|u| + |z_k|} \geq \frac{4r - r}{r + r} = \frac{3}{2}.$$

Thus $S_1 \geq n \log \frac{3}{2}$.

For $k \geq n + 1$ we have $|z_k| \geq 8r > |v|$ so that

$$\log \frac{|v - z_k|}{|u - z_k|} \geq \log \frac{|z_k| - |v|}{|u| + |z_k|} = \log \left(1 - \frac{4r}{|z_k|}\right) - \log \left(1 + \frac{r}{|z_k|}\right) \geq -\frac{8r}{|z_k|} - \frac{r}{|z_k|}.$$

Here we have used the inequalities $\log(1 + x) \leq x$ and $\log(1 - x) \geq -2x$ valid for $0 \leq x \leq \frac{1}{2}$. For large n we also have $|z_{k+1}/z_k| \geq 2$ if $k \geq n + 1$. We find that

$$\log \frac{|v - z_k|}{|u - z_k|} \geq -\frac{9r}{|z_k|} \geq -\frac{9r}{|z_{n+1}|} 2^{n+1-k} \geq -\frac{9}{8} 2^{n+1-k}$$

for $k \geq n + 1$. It follows that

$$S_2 \geq -\frac{9}{8} \sum_{k=n+1}^{\infty} 2^{n+1-k} = -\frac{9}{4}.$$

Together with the estimate for S_1 this implies that (4.1) holds for large n.

Let now U be the component of $g^{-1}(B(0, M(r, g)))$ which contains $B(0, r)$. It follows from (4.1) that $U \subset B(0, 4r)$. By our choice of r the number of zeros of g in U is n. The Riemann-Hurwitz formula yields that g' has $n - 1$ zeros in U. Thus g' has at most $n - 1$ zeros in $B(0, r)$ and at least $n - 1$ zeros in $B(0, 4r)$. Thus $r \leq |z'_n|$

and $|z'_{n-1}| \leq 4r$. Since this holds for any r satisfying $|z_n| \leq r$ and $8r \leq |z_{n+1}|$ we conclude, choosing $r = |z_n|$ or $r = \frac{1}{8}|z_{n+1}|$, that $|z'_{n-1}| \leq 4|z_n|$ and $\frac{1}{8}|z_{n+1}| \leq |z'_n|$ for large n. Thus $\frac{1}{8}|z_{n+1}| \leq |z'_n| \leq 4|z_{n+1}|$ for large n. The conclusion follows. □

The following result is a variant of Lemma 3.7.

Lemma 4.6. *Let f and g be entire transcendental functions. Suppose that there exists a sequence (ξ_n) such that $(f \circ g)(\xi_n) = \xi_n$ and $(f \circ g)'(\xi_n) \to \infty$. Then $\eta_n := g(\xi_n)$ satisfies $(g \circ f)(\eta_n) = \eta_n$ and $(g \circ f)'(\eta_n) = (f \circ g)'(\xi_n) \to \infty$.*

The proof is straightforward and thus omitted.

4.2. **Proof of Theorem 2.** We proceed as in the proof of Theorem A and define the sequences (c_n), (r_n), (f_n) and (g_n) as there. Again we find that (g_n) is not quasinormal in $B(0, 2)$. In the proof of Theorem A we noted that by passing to a subsequence we can achieve that no subsequence of (g_n) is normal at any of three points $b_1, b_2, b_3 \in \mathbb{C}\backslash\{0\}$. The same argument yields this for any number of points b_k. Moreover, we can achieve that these points are in $B(0, 2)\backslash\{0\}$.

We will have to distinguish several cases now.

Case 1. It is possible to choose the sequence (c_n) such that (f_n) is not quasinormal of order 2.

Passing to a subsequence if necessary we may assume that there exist $a_1, a_2, a_3 \in \mathbb{C}$ where no subsequence of (f_n) is normal and that there exist $b_1, \dots, b_7 \in \mathbb{C}\backslash\{0\}$ where no subsequence of (g_n) is normal.

We choose $0 < \varepsilon < \frac{1}{2}$ such that the closures of the disks $B(a_j, \varepsilon)$ are pairwise disjoint. Moreover, we require that the closures of the disks $B(b_j, \varepsilon)$ are pairwise disjoint and do not contain 0. We also choose $0 < \delta < \frac{\varepsilon}{2}$.

It follows from Lemma 4.1 that if n is sufficiently large and $j \in \{1, 2, 3\}$, then f_n has a simple island in $B(a_j, \varepsilon)$ over at least five of the seven disks $B(b_k, \varepsilon)$. Overall we obtain at least 15 domains in the union of the three disks $B(a_j, \varepsilon)$ which are simple islands over one of the seven disks $B(b_k, \varepsilon)$. This implies that there exists $k \in \{1, \dots, 7\}$ such that f_n has three simple islands $U_j \subset B(a_j, \varepsilon)$, $j \in \{1, 2, 3\}$, over the same disk $B(b_k, \varepsilon)$. It also follows from Lemma 2.1 that if n is sufficiently large, then there exists $j \in \{1, 2, 3\}$ such that g_n has a simple island $V \subset B(b_k, \delta)$ over $B(a_j, \varepsilon)$. Then $W := V \cap g_n^{-1}(U_j)$ is a simple island of $f_n \circ g_n$ over $B(b_k, \varepsilon)$, and $W \subset B(b_k, \delta)$. Lemma 4.4 implies that $f_n \circ g_n$ has a fixed point $\xi \in W$ with $|(f_n \circ g_n)'(\xi)| \geq \varepsilon/4\delta$. Then $\xi_n := r_n\xi$ is a fixed point of $f \circ g$ with $|(f \circ g)'(\xi_n)| \geq \varepsilon/4\delta$. Since δ can be chosen arbitrarily small the conclusion follows.

Case 2. For every choice of the sequence (c_n) the sequence (f_n) is quasinormal of order 2.

Then $f_n \to \infty$ in $\mathbb{C}\backslash\{0, 1\}$ by Lemma 2.2. Let $K > 2$. For sufficiently large n we then have $|f_n(z)| > 2$ if $\frac{1}{4} \leq |z| \leq K$ and $|z - 1| \geq \frac{1}{4}$.

We now distinguish two subcases.

Case 2.1. There are infinitely many n such that $B(1, \frac{1}{4})$ contains at least two zeros of f_n.

Passing to a subsequence if necessary we may assume that this holds for all n. We note that if n is sufficiently large, then $B(0, \frac{1}{4})$ also contains at least two zeros of f_n. With $a_1 := 0$ and $a_2 := 1$ thus both disks $B(a_j, \frac{1}{4})$ contain at least two zeros of f_n.

We may assume that no subsequence of (g_n) is normal at five points $b_1, \dots, b_5 \in B(0, 2)\backslash\{0\}$. Again we choose $0 < \varepsilon < \frac{1}{2}$ such that the closures of the disks $B(b_k, \varepsilon)$

are pairwise disjoint and do not contain 0, and we choose $0 < \delta < \frac{\varepsilon}{2}$. We can now deduce from Lemma 2.1 that if n is sufficiently large and $k \in \{1, \ldots, 5\}$, then g_n has an island over one of the disks $B(a_j, \frac{1}{4})$ in $B(b_k, \delta)$. It follows that for at least three values of k the function g_n has an island over the same disk $B(a_j, \frac{1}{4})$ in $B(b_k, \delta)$. We may assume that the b_k are numbered such that this holds for $k \in \{1, 2, 3\}$.

We claim that there exists $k \in \{1, 2, 3\}$ such that f_n has two simple islands U_1, U_2 in $B(a_j, \frac{1}{4})$ over $B(b_k, \varepsilon)$. To this end we assume first that there are two components X_1, X_2 of $f_n^{-1}(B(0,2))$ contained in $B(a_j, \frac{1}{4})$ in which f_n is univalent. Then we can simply take $U_\ell := f_n^{-1}(B(b_k, \varepsilon)) \cap X_\ell$, for $\ell \in \{1, 2\}$ and arbitrary $k \in \{1, 2, 3\}$. Suppose now that such components X_1, X_2 do not exist. Since $|f_n(z)| > 2$ if $|z - a_j| = \frac{1}{4}$ and since f_n has at least two zeros in $B(a_j, \frac{1}{4})$, there now exists a component X of $f_n^{-1}(B(0,2))$ contained in $B(a_j, \frac{1}{4})$ such that $f : X \to B(0,2)$ is a proper map of degree at least 2. Now Lemma 4.3 yields our claim that there exists $k \in \{1, 2, 3\}$ such that f_n has two simple islands $U_1, U_2 \subset X \subset B(a_j, \frac{1}{4})$ over $B(b_k, \varepsilon)$.

Recall that in turn g_n has an island $V \subset B(b_k, \delta)$ over $B(a_j, \frac{1}{4})$. Lemma 4.3, applied to the proper map $g_n : V \to B(a_j, \frac{1}{4})$, now implies that V contains a domain W which is a simple island of g_n over U_1 or U_2. Then W is a simple island of $f_n \circ g_n$ over $B(b_k, \varepsilon)$, and $W \subset V \subset B(b_k, \delta)$. As before Lemma 4.4 implies that $f_n \circ g_n$ has a fixed point $\xi \in W$ with $|(f_n \circ g_n)'(\xi)| \geq \varepsilon/4\delta$. Again $\xi_n := r_n \xi$ is a fixed point of $f \circ g$ with $|(f \circ g)'(\xi_n)| \geq \varepsilon/4\delta$, and the conclusion follows since δ can be chosen arbitrarily small.

Case 2.2. For all sufficiently large n the disk $B(1, \frac{1}{4})$ contains at most one zero of f_n.

Since $|f_n(z)| > 2$ for $\frac{1}{4} \leq |z| \leq K$ and $|z - 1| \geq \frac{1}{4}$, provided n is sufficiently large, we conclude that the annulus $R := \{z \in \mathbb{C} : \frac{1}{4} \leq |z| \leq K\}$ contains at most one zero of f_n. On the other hand, since $f_n(1) \to 0$ we conclude that $B(1, \frac{1}{4})$ contains a component Y of $f_n^{-1}(B(0,2))$ and thus in particular a zero of f_n for large n. Thus f_n has exactly one zero in R if n is sufficiently large. In fact, f_n takes every value in $B(0,2)$ exactly once in Y, and since there are no other components of $f_n^{-1}(B(0,2))$ intersecting R, we see that f_n takes every value in $B(0,2)$ exactly once in R. We find that if $z_1 \in Y$ and $z_2 \in \mathbb{C}$ with $|z_2| \geq |z_1|$ and $f_n(z_2) = f_n(z_1) \in B(0,2)$, then $|z_2| \geq K$ while $|z_1| \leq \frac{5}{4}$ so that $|z_2| \geq \frac{4}{5}K|z_1|$.

In terms of f we see that f has infinitely many c-points for every $c \in \mathbb{C}\backslash\{0\}$, and the sequence (w_n) of c-points of f, arranged such that $|w_{n+1}| \geq |w_n|$, satisfies $|w_{n+1}| \geq \frac{4}{5}K|w_n|$. Since K can be chosen arbitrarily large we deduce that $\lim_{n\to\infty} |w_{n+1}/w_n| = \infty$. It follows that the number $n(r, 1/(f - c))$ of c-points of f in $\overline{B(0,r)}$ satisfies $n(r, 1/(f - c)) = o(\log r)$ as $r \to \infty$, for any $c \in \mathbb{C}\backslash\{0\}$. Standard estimates from value distribution theory [17, 18, 21] now imply that $\log M(r, f) = o((\log r)^2)$ as $r \to \infty$. In particular, f has order 0.

Lemma 4.6 says that the conclusion is symmetric with respect to f and g. We may thus assume that interchanging the roles of f and g leads again to Case 2.2. We find that g is also of order 0 and that the sequence (z_n) of zeros of g satisfies $\liminf_{n\to\infty} |z_{n+1}/z_n| = \infty$.

Assuming without loss of generality that $g(0) \neq 0$ we deduce from Lemma 4.5 that the sequence (z_n') of zeros of g' satisfies

$$\lim_{n\to\infty} \frac{|z_{n+1}'|}{|z_n'|} = \infty. \tag{4.2}$$

Proceeding as before we may assume, passing to a subsequence if necessary, that no subsequence of (g_n) is normal at four points $b_1, \dots, b_4 \in B(0,2)\backslash\{0\}$. As before we choose $0 < \varepsilon < \frac{1}{2}$ such that the closures of the disks $B(b_j, \varepsilon)$ are pairwise disjoint and do not contain 0, and we choose $0 < \delta < \frac{\varepsilon}{2}$.

With $a_1 := 0$ and $a_2 := 1$ we can deduce as before from Lemma 2.1 that if n is sufficiently large and $k \in \{1, \dots, 4\}$, then g_n has an island $V_k \subset B(b_k, \delta)$ over one of the disks $B(a_j, \frac{1}{4})$. It follows from (4.2) that at most one of these four islands contains a zero of g_n', provided n is large enough. Thus V_k is a simple island for at least three values of k. Hence there exist two values of k such that V_k is a simple island over the same disk $B(a_j, \frac{1}{4})$. We may assume that the b_k are numbered such that this holds for $k \in \{1, 2\}$. By Lemma 4.2 the function f_n has a simple island $U \subset B(a_j, \frac{1}{4})$ over $B(b_1, \varepsilon)$ or $B(b_2, \varepsilon)$. Without loss of generality we can assume that U is a simple island over $B(b_1, \varepsilon)$. Then $W := V_1 \cap g_n^{-1}(U)$ is a simple island of $f_n \circ g_n$ over $B(b_1, \varepsilon)$, and $W \subset B(b_1, \delta)$. As before we deduce that $f_n \circ g_n$ has a fixed point $\xi \in W$ with $|(f_n \circ g_n)'(\xi)| \geq \varepsilon/4\delta$. Again $\xi_n := r_n \xi$ is a fixed point of $f \circ g$ with $|(f \circ g)'(\xi_n)| \geq \varepsilon/4\delta$ so that the conclusion follows also in this case. $\square$

5. Proof of Theorem 3

5.1. **Preliminary Lemmas.** We shall need the following quantitative version of Lemma 2.1.

Lemma 5.1. *Let $D_1, D_2 \subset \mathbb{C}$ be Jordan domains with disjoint closures and let $f : B(a, r) \to \mathbb{C}$ be a holomorphic function which has no island over D_1 or D_2. Then*

$$\frac{|f'(a)|}{2\mu(\log \mu + A)} \leq \frac{1}{r}$$

where $\mu = \max\{1, |f(a)|\}$ and A is a constant depending only on the domains D_1 and D_2.

While Lemma 2.1 says that the family of holomorphic functions having no islands over any of two given domains is normal, Lemma 5.1 is based on the fact that this family is in fact a normal *invariant* family. Lemma 5.1 follows from Lemma 2.1 together with results of Hayman [16] on normal invariant families. It is a direct consequence of Theorems 6.8, 6.6, and 5.5 of his book [17].

We also need the following version of a classical growth lemma due to Borel.

Lemma 5.2. *Let $r_0 > 0$ and let $T : [r_0, \infty) \to [e, \infty)$ be increasing and continuous. Define*

$$F := \left\{ r \geq r_0 : T\left(r\left(1 + \frac{1}{(\log T(r))^2}\right)\right) > eT(r) \right\}.$$

Then

$$\int_F \frac{dt}{t} \leq \frac{\pi^2}{6}.$$

Proof. We may assume that $F \neq \emptyset$ and define

$$r_1 := \inf F, \quad r_1' := r_1 \left(1 + \frac{1}{(\log T(r_1))^2}\right)$$

and then inductively

$$r_k := \inf\left(F \cap [r_{k-1}', \infty)\right), \quad r_k' := r_k \left(1 + \frac{1}{(\log T(r_k))^2}\right).$$

If $F \cap [r_k', \infty) = \emptyset$ for some k so that the process terminates, then we put $N := k$. Otherwise we put $N := \infty$.

We have $T(r_k) \geq T(r_{k-1}') \geq eT(r_{k-1})$. Hence $T(r_k) \geq e^{k-1}T(r_1) \geq e^k$ so that $(\log T(r_k))^2 \geq k^2$. If $N = \infty$ we thus have $r_k \to \infty$ as $k \to \infty$. In any case we find that

$$F \subset \bigcup_{k=1}^{N} [r_k, r_k']$$

so that

$$\int_F \frac{dt}{t} \leq \sum_{k=1}^{N} \log \frac{r_k'}{r_k} = \sum_{k=1}^{N} \log\left(1 + \frac{1}{(\log T(r_k))^2}\right).$$

Since $\log(1+x) < x$ for $x > 0$ this yields

$$\int_F \frac{dt}{t} \leq \sum_{k=1}^{N} \frac{1}{(\log T(r_k))^2} \leq \sum_{k=1}^{\infty} \frac{1}{k^2} = \frac{\pi^2}{6}.$$

□

5.2. **Proof of Theorem 3.** We proceed as in the proof of Theorem A and define the sequences (c_n), (r_n), (f_n) and (g_n) and the points a_1, a_2 and b_1, b_2, b_3 as there. If we could choose all three points b_1, b_2, b_3 nonreal, then the argument given there would imply that $f \circ g$ has infinitely many nonreal fixed points. We may thus assume that (g_n) is quasinormal of order 2 in $\mathbb{C} \backslash \mathbb{R}$ and thus $g_n \to 0$ locally uniformly in $\mathbb{C} \backslash \mathbb{R}$ by Lemma 2.2.

If there exist three annuli $\Omega_j := \{z \in \mathbb{C} : S_j < |z| < T_j\}$ with disjoint closures such that g_n has an island over $B(a_1, \varepsilon)$ or $B(a_2, \varepsilon)$ in $\Omega_j \backslash \mathbb{R}$, for all $j \in \{1, 2, 3\}$ and infinitely many n, then the argument used in the proof of Theorem A shows again that $f_n \circ g_n$ has a fixed point in one of these islands, and thus $f \circ g$ has infinitely many nonreal fixed points.

We may thus assume that such annuli Ω_j do not exist. Passing to a subsequence if necessary we thus find an annulus $\Omega := \{z \in \mathbb{C} : S - 1 < |z| < T + 1\}$ with $S > 1$ and $T > e^2 S$ such that no g_n has an island over $B(a_1, \varepsilon)$ or $B(a_2, \varepsilon)$ in $\Omega \backslash \mathbb{R}$. Since $g_n \to 0$ in $\mathbb{C} \backslash \mathbb{R}$ we may assume that $|g_n(z)| \leq 1$ for $S \leq |z| \leq T$ and $|\operatorname{Im} z| \geq 1$.

To save indices, we now write $h := g_n$. For $S \leq |z| \leq T$ and $|\operatorname{Im} z| \leq 1$ we conclude from Lemma 5.1 that

$$\frac{|h'(z)|}{2\mu(\log \mu + A)} \leq \frac{1}{|\operatorname{Im} z|}$$

where $\mu = \max\{1, |h(z)|\}$ and A is a constant. We may assume that $A > 1$. If $S \leq r \leq T$ and $\log |h(re^{it})| \geq A$, then $|h(re^{it})| \geq e^A > 1$ so that $|\operatorname{Im}(re^{it})| =$

$r|\sin t| \le 1$ and thus

$$\frac{|h'(re^{it})|}{|h(re^{it})|\log|h(re^{it})|} \le \frac{2|h'(re^{it})|}{|h(re^{it})|(\log|h(re^{it})|+A)} \le \frac{4}{r|\sin t|}. \tag{5.1}$$

We denote by $T(r,h)$ the Nevanlinna characteristic of h and recall the inequality

$$\log M(r,h) \le \frac{R+r}{R-r}T(R,h) \tag{5.2}$$

valid for $0 < r < R$. Since $M(r,g_n) \to \infty$ as $n \to \infty$ if $r > 1$ we conclude that $T(r,h) = T(r,g_n) \to \infty$ for $r > 1$. In particular we may assume that $T(r,h) \ge 3A > 3$ for $r \ge S$.

Choosing

$$R := r\left(1 + \frac{1}{(\log T(r))^2}\right)$$

in (5.2) and noting that $\log(T/S) > 2 > \pi^2/6$ we deduce from Lemma 5.2 that there exists $r \in [S,T]$ such that

$$\log M(r,h) \le \left(2 + \frac{1}{(\log T(r,h))^2}\right)(\log T(r,h))^2 eT(r,h).$$

Now $(\log T(r,h))^2 \ge (\log 3A)^2 \ge 1$ and hence

$$\log M(r,h) \le 3eT(r,h)(\log T(r,h))^2. \tag{5.3}$$

For a value of r satisfying (5.3) we consider the set

$$E(r) := \left\{t \in [0,2\pi] : \log|h(re^{it})| \ge \frac{1}{2}T(r,h)\right\}$$

and define $\lambda(r) := \operatorname{meas} E(r)$, where $\operatorname{meas} E$ denotes the measure of a set E. Then for at least one the four sets $E_\ell(r) := E(r) \cap [\ell\frac{\pi}{2}, (\ell+1)\frac{\pi}{2}]$, where $\ell \in \{0,1,2,3\}$, we have $\operatorname{meas} E_\ell(r) \ge \frac{1}{4}\lambda(r)$. We assume now that this is the case for $\ell = 0$. The modifications that have to be made for the other cases will be obvious. We define

$$\alpha(r) := \max E_0(r)$$

and

$$\beta(r) := \min\left\{t \in \left[\alpha(r), \frac{\pi}{2}\right] : \log|h(re^{it})| = A\right\}$$

Then $\alpha(r) \ge \operatorname{meas} E_0(r) \ge \frac{1}{4}\lambda(r)$. For $\alpha(r) \le t \le \beta(r)$ we have $1 < A \le \log|h(re^{it})| \le \frac{1}{2}T(r,h)$. Thus $\log\log h$ may be defined on the arc $\{re^{it} : \alpha(r) \le t \le \beta(r)\}$, and we may choose the branch of the logarithm such that $|\log h(re^{i\beta(r)})| \le \log|h(re^{i\beta(r)})| + \pi = A + \pi$ and hence $\log|\log h(re^{i\beta(r)})| \le \log(A+\pi)$. Thus

$$\begin{aligned}
\log\log|h(re^{i\alpha(r)})| - \log(A+\pi) &\le \log|\log h(re^{i\alpha(r)})| - \log|\log h(re^{i\beta(r)})| \\
&\le |\log\log h(re^{i\alpha(r)}) - \log\log h(re^{i\beta(r)})| \\
&= \left|\int_{\alpha(r)}^{\beta(r)} \frac{h'(re^{it})|}{h(re^{it})\log h(re^{it})} ire^{it}dt\right| \\
&\le r\int_{\alpha(r)}^{\beta(r)} \frac{|h'(re^{it})|}{|h(re^{it})|\log|h(re^{it})|}dt \\
&\le 4\int_{\alpha(r)}^{\beta(r)} \frac{dt}{|\sin t|}.
\end{aligned}$$

Here the last inequality follows from (5.1). Now $\sin t \geq 2t/\pi$ for $0 \leq t \leq \pi/2$. Thus

$$\log\log|h(re^{i\alpha(r)})| - \log(A+\pi) \leq 2\pi \int_{\alpha(r)}^{\beta(r)} \frac{dt}{t} = 2\pi \log\frac{\beta(r)}{\alpha(r)} \leq 2\pi \log\frac{\pi}{2\alpha(r)}.$$

Since

$$\log|h(re^{i\alpha(r)})| = \frac{1}{2}T(r,h)$$

this yields

$$\log T(r,h) \leq 2\pi \log\frac{\pi}{2\alpha(r)} + \log(A+\pi) + \log 2. \tag{5.4}$$

Now

$$\begin{aligned} T(r,h) &= \frac{1}{2\pi}\int_0^{2\pi} \log^+|h(re^{it})|dt \\ &= \frac{1}{2\pi}\int_{E(r)} \log^+|h(re^{it})|dt + \frac{1}{2\pi}\int_{[0,2\pi]\setminus E(r)} \log^+|h(re^{it})|dt \\ &\leq \frac{1}{2\pi}\lambda(r)\log M(r,h) + \frac{1}{2}T(r,h) \end{aligned}$$

so that

$$\frac{\lambda(r)}{\pi} \geq \frac{T(r,h)}{\log M(r,h)}.$$

Hence

$$\frac{\pi}{2\alpha(r)} \leq \frac{2\pi}{\lambda(r)} \leq \frac{2\log M(r,h)}{T(r,h)}.$$

Using (5.3) we find that

$$\frac{\pi}{2\alpha(r)} \leq 6e(\log T(r,h))^2.$$

Together with (5.4) this yields

$$\log T(r,h) \leq 2\pi \log\left(6e(\log T(r,h))^2\right) + \log(A+\pi) + \log 2.$$

This implies that

$$\log T(r,h) \leq 4\pi \log\log T(r,h) + C$$

for some constant C, which is a contradiction since $T(r,h) = T(r,g_n) \to \infty$ as $n \to \infty$. □

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Mathematisches Seminar, Christian–Albrechts–Universität zu Kiel, Ludewig–Meyn–Str. 4, D–24098 Kiel, Germany

E-mail address: bergweiler@math.uni-kiel.de