

QUASINORMAL FAMILIES AND PERIODIC POINTS

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Dedicated to Larry Zalcman on his 60th Birthday

ABSTRACT. Let $n \geq 2$ be an integer and $K > 1$. By f^n we denote the n -th iterate of a function f . Let \mathcal{F} be the family of all functions f holomorphic in some domain such that $|(f^n)'(\xi)| \leq K$ whenever $f^n(\xi) = \xi$. We show that \mathcal{F} is quasinormal of order 1. If K is sufficiently small, then \mathcal{F} is normal. We also show that if f is a transcendental entire function, then f has a sequence (ξ_k) of periodic points of period n such that $(f^n)'(\xi_k) \rightarrow \infty$ as $k \rightarrow \infty$.

1. INTRODUCTION AND MAIN RESULTS

Let $D \subset \mathbb{C}$ be a domain and let $f : D \rightarrow \mathbb{C}$ be a holomorphic function. The iterates $f^n : D_n \rightarrow \mathbb{C}$ of f are defined by $D_1 := D$, $f^1 := f$ and $D_n := f^{-1}(D_{n-1})$, $f^n := f^{n-1} \circ f$ for $n \in \mathbb{N}$, $n \geq 2$. Note that $D_2 = f^{-1}(D_1) \subset D = D_1$ and thus $D_{n+1} \subset D_n \subset D$ for all $n \in \mathbb{N}$.

A point $\xi \in D$ is called a *periodic point of period n* of f if $\xi \in D_n$ and $f^n(\xi) = \xi$, but $f^m(\xi) \neq \xi$ for $1 \leq m \leq n - 1$. A periodic point of period 1 is called a *fixed point*. The periodic points of period n are thus the fixed points of f^n which are not fixed points of f^m for any m less than n . Let ξ be a periodic point of period n of f . We say that ξ is *repelling* if $|(f^n)'(\xi)| > 1$.

The periodic points play an important role in complex dynamics. For example, the Julia set of a rational or entire function, which is defined as the set where the iterates fail to be normal, is the closure of the set of repelling periodic points.

The following result is due to M. Essén and S. Wu [11, Theorem 1].

Theorem A. *Let $D \subset \mathbb{C}$ be a domain and let \mathcal{F} be the family of all holomorphic functions $f : D \rightarrow \mathbb{C}$ for which there exists $n = n(f) > 1$ such that f^n has no repelling fixed point. Then \mathcal{F} is normal.*

Without the word “repelling” the same authors had proved this result earlier in [10]. We mention that the results by Essén and Wu [10, 11] answered a question of L. Yang [20, Problem 8].

We note that in Theorem A the condition that f^n has no repelling fixed point cannot be replaced by the condition that f has no periodic point of period n . In fact, the family $\mathcal{F} = \{nz\}_{n \in \mathbb{N}}$ is not normal at 0, and the functions in \mathcal{F} do not have periodic points of period greater than one. Also, $\mathcal{F} = \{-z + az^2\}_{a \in \mathbb{C} \setminus \{0\}}$ is not normal at 0, and the functions in \mathcal{F} do not have periodic points of period 2. It was shown in [3, Theorem 2] that – in a suitable sense – non-normal sequences of holomorphic functions which fail to have periodic points of some period greater than one always arise from these examples.

Moreover, the following result was proved in [3, Theorem 3].

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Theorem B. *Let $D \subset \mathbb{C}$ be a domain and let \mathcal{F} be the family of all holomorphic functions $f : D \rightarrow \mathbb{C}$ for which there exists $n = n(f) > 1$ such that f has no repelling periodic point of period n . Then \mathcal{F} is quasinormal.*

Recall here that a family \mathcal{F} of functions holomorphic in a domain D is called *quasinormal* (cf. [8, 14, 19]) if for each sequence (f_k) in \mathcal{F} there exists a subsequence (f_{k_j}) and a finite set $E \subset D$ such that (f_{k_j}) converges locally uniformly in $D \setminus E$. If the cardinality of the exceptional set E can be bounded independently of the sequence (f_k) , and if q is the smallest such bound, then we say that \mathcal{F} is quasinormal of order q .

Our first result is an improvement of Theorem B.

Theorem 1.1. *Let $K > 1$ and let $D \subset \mathbb{C}$ be a domain. Let \mathcal{F} be the family of all holomorphic functions $f : D \rightarrow \mathbb{C}$ for which there exists $n = n(f) > 1$ such that $|(f^n)'(\xi)| \leq K$ for every periodic point ξ of period n of f . Then \mathcal{F} is quasinormal of order 1.*

The following results are closely connected to Theorems A and B.

Theorem C. *Let f be a transcendental entire function and let $n \in \mathbb{N}$, $n \geq 2$. Then f^n has infinitely many fixed points.*

Theorem D. *Let f be a transcendental entire function and let $n \in \mathbb{N}$, $n \geq 2$. Then f has infinitely many repelling periodic points of period n .*

Theorem C is due to P. C. Rosenbloom [18] while Theorem D can be found in [4, Theorem 1]. We mention that already P. Fatou [12, p. 345] had proved that the second iterate of a transcendental entire function has a fixed point; he had used this result to show [12, p. 348–350] that the Julia set of an entire transcendental function is always non-empty.

The connection between Theorems A and B on the one hand and Theorems C and D on the other hand is given by a heuristic principle attributed to A. Bloch which relates normal families and entire functions. This principle says that the family of all holomorphic functions with a certain property is likely to be normal if all entire functions with this property are constant. More generally, one may expect normality or at least quasinormality, if there are only “few” entire functions with this property. For an excellent discussion of Bloch’s principle we refer to the book by J. Schiff [19] and two papers by L. Zalcman [21, 22].

The following sharpening of Theorem D can be considered as an analogue of Theorem 1.1 for entire functions according to Bloch’s principle.

Theorem 1.2. *Let f be a transcendental entire function and let $n \in \mathbb{N}$, $n \geq 2$. Then f has a sequence (ξ_k) of periodic points of period n such that $(f^n)'(\xi_k) \rightarrow \infty$ as $k \rightarrow \infty$.*

Theorem 1.1 shows in particular that if $K > 1$, then the family \mathcal{F}_K of all holomorphic functions f for which there exists $n \geq 2$ such that $|(f^n)'(\xi)| \leq K$ for all fixed points ξ of f^n is quasinormal. Considering the family $\mathcal{G} = \{az^2\}_{a \in \mathbb{C} \setminus \{0\}}$ we see that \mathcal{F}_K fails to be normal if $K \geq 4$. Indeed, for $f(z) = az^2 \in \mathcal{G}$ the fixed points of f^2 are given by $\xi = 0$ and $\xi = \omega/a$ where ω is a third root of unity, and we have $(f^2)'(0) = 0$ and $(f^2)'(\omega/a) = 4$ so that $\mathcal{G} \subset \mathcal{F}_4$. Clearly \mathcal{G} is not normal at 0, and thus \mathcal{F}_K is not normal if $K \geq 4$.

It does not seem unlikely that \mathcal{F}_K is normal if $K < 4$. In this direction we have the following result.

Theorem 1.3. *For each integer n greater than 1 there exists a constant $K_n > 1$ with the following property: if $D \subset \mathbb{C}$ is a domain and \mathcal{F} is a family of holomorphic functions $f : D \rightarrow \mathbb{C}$ such that $|(f^n)'(\xi)| \leq K_n$ for all fixed points ξ of f^n , then \mathcal{F} is normal.*

The example mentioned above shows that the conclusion of Theorem 1.3 does not hold for $K_n = 2^n$. Possibly it holds for each $K_n < 2^n$.

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2. PRELIMINARY LEMMAS

We shall build on the ideas developed in [3, 10, 11]. As in these papers, one of the central tools comes from the Ahlfors theory of covering surfaces; see [1], [13, Chapter 5] or [15, Chapter XIII] for an account of this theory. The idea to use the Ahlfors theory to prove the existence of repelling periodic points is due to I. N. Baker [2] who had used it to show that such points are dense in the Julia set of an entire transcendental function. For a survey of further applications of the Ahlfors theory in complex dynamics we refer to [6].

Given a holomorphic function $f : D \rightarrow \mathbb{C}$ and a Jordan domain $V \subset \mathbb{C}$, we say that f has a *simple island* over V if $f^{-1}(V)$ has a component U with $\overline{U} \subset D$ such that $f|_U : U \rightarrow V$ is bijective. Such a component U is then called a *simple island* over V . We will often use the trivial observation that if f has a simple island U over V and if V' is a Jordan domain contained in V , then f has a simple island $U' \subset U$ over V' .

The result from the Ahlfors theory that we need is the following Lemma 2.1. Besides the references already mentioned, we refer to [5] for a proof.

Lemma 2.1. *Let $D \subset \mathbb{C}$ be a domain and let D_1, D_2 and D_3 be Jordan domains with pairwise disjoint closures. Let \mathcal{F} be a family of functions holomorphic in D which is not normal. Then there exists a function $f \in \mathcal{F}$ which has an island over D_1, D_2 or D_3 .*

We mention that this result has a counterpart for entire functions according to the principle by Bloch already mentioned: if f is a non-constant entire function and if D_1, D_2, D_3 are Jordan domains with pairwise disjoint closures, then f has a simple island over at least one of these domains. This result (and similarly Lemma 2.1) does not hold if we take only two domains D_1 and D_2 , as shown by the example $f(z) = \cos z$, $D_1 = D(-1, \frac{1}{2})$, $D_2 = D(1, \frac{1}{2})$, with $D(a, r) := \{z \in \mathbb{C} : |z - a| < r\}$. A rather simple, but for the purposes of this paper quite important observation is that if f is a polynomial, then we need only two domains D_1 and D_2 in the above statement. More generally, this holds for polynomial-like mappings f .

By definition, if $U, V \subset \mathbb{C}$ are Jordan domains with $\overline{U} \subset V$, and if $f : U \rightarrow V$ is a proper holomorphic map (of degree d), then the triple (f, U, V) is called a *polynomial-like map* (of degree d). The basic result about polynomial-like maps (see [7, Theorem VI.1.1] or [9, Theorem 1]) says that polynomial-like maps are quasiconformally conjugate to polynomials (of the same degree), but we shall not need this result. We mention that the concept of a polynomial-like mapping, created by A. Douady and J. H. Hubbard [9], was also used in [3, 4, 10, 11] to prove the existence of (repelling) fixed points and periodic points.

Lemma 2.2. *Let (f, U, V) be a polynomial-like map of degree d and let D_1 and D_2 be Jordan domains with pairwise disjoint closures contained in V . Then there exist two domains $U_1, U_2 \subset U$ which are simple islands over D_1 or D_2 .*

Note that U_1 and U_2 need not be islands over the same domain. We allow the possibility that U_1 is an island over D_1 and U_2 is an island over D_2 , or vice versa.

Proof of Lemma 2.2. We denote by V_1, \dots, V_m the components of $f^{-1}(D_1)$ and by W_1, \dots, W_n the components of $f^{-1}(D_2)$. Let μ_j be the degree of the proper map $f|_{V_j} : V_j \rightarrow D_1$ and let ν_j be the degree of the proper map $f|_{W_j} : W_j \rightarrow D_2$. Then $d = \sum_{j=1}^m \mu_j = \sum_{j=1}^n \nu_j$. By the Riemann-Hurwitz formula, f has $d - 1$ critical points in U , and of these critical points there are $\mu_j - 1$ critical points in V_j and $\nu_j - 1$ critical points in W_j . Thus

$$d - 1 \geq \sum_{j=1}^m (\mu_j - 1) + \sum_{j=1}^n (\nu_j - 1) = \sum_{j=1}^m \mu_j + \sum_{j=1}^n \nu_j - m - n = 2d - m - n.$$

This yields $m + n \geq d + 1$. Since f has $d - 1$ critical points, at least two of the domains V_j, W_j do not contain a critical point and thus are simple islands.

The following lemma is simple and well-known.

Lemma 2.3. *Let $0 < \delta < \frac{\varepsilon}{2}$ and let $U \subset D(a, \delta)$ be a simply-connected domain. Let $f : U \rightarrow D(a, \varepsilon)$ be holomorphic and bijective. Then f has a fixed point ξ in U which satisfies $|f'(\xi)| \geq \varepsilon/4\delta$.*

Proof. We consider the inverse function $g : D(a, \varepsilon) \rightarrow U$ of f . It follows easily from Rouché's theorem that g has a fixed point $\xi \in U \subset D(a, \delta)$. The function

$$h(z) = \frac{1}{2\delta} \left(g \left(\frac{\varepsilon}{2} z + \xi \right) - \xi \right)$$

then maps the unit disk $D(0, 1)$ into itself and satisfies $h(0) = 0$. From Schwarz's lemma we can now deduce that $1 \geq |h'(0)| = \varepsilon|g'(\xi)|/4\delta$. The conclusion follows since $f'(\xi) = 1/g'(\xi)$.

We recall some elementary graph theoretic notions used in [3] (and implicit in [10, 11]). For a set V and a set $E \subset V \times V$ we call the pair $G = (V, E)$ a digraph. The elements of V are called vertices and those of E are called edges. In contrast to the usual terminology we allow loops; that is, we do not exclude edges e of the form $e = (v, v)$ with $v \in V$.

We call $w = (v_0, v_1, \dots, v_n) \in V^{n+1}$ a *closed walk of length n* if $v_0 = v_n$ and $(v_{k-1}, v_k) \in E$ for $k = 1, \dots, n$. Note that we do not assume that $v_j \neq v_k$ for $0 \leq j < k \leq n - 1$. We call a closed walk $w = (v_0, v_1, \dots, v_n)$ *primitive* if there does not exist $p \in \mathbb{N}$, $1 \leq p < n$, such that $p|n$ and $v_j = v_k$ for all j, k satisfying $p|(j - k)$. A primitive closed walk is thus a closed walk which is not obtained by running through a closed walk of smaller length several times. Finally recall that the outdegree of a vertex v is defined to be the cardinality of $\{u \in V : (v, u) \in E\}$.

As in [3, §5] we shall use the following lemma.

Lemma 2.4. *Let $q, n \in \mathbb{N}$, $q \geq 6$, $n \geq 2$. Let G be a digraph with q vertices such that the outdegree of each vertex is at least $q - 2$. Then G contains a primitive closed walk of length n .*

We shall also use the following well-known result (see [8, p. 131] or [19, Proposition A.2]) about quasinormal families, whose simple proof we include for completeness.

Lemma 2.5. *Let \mathcal{F} be a family of functions holomorphic in a domain D . Suppose that \mathcal{F} is quasinormal of order q . Let (f_k) be a sequence in \mathcal{F} and let $a_1, \dots, a_q \in D$. If no subsequence of (f_k) is normal at any of the points a_j , then $f_k \rightarrow \infty$ in $D \setminus \{a_1, \dots, a_q\}$.*

Proof. Since \mathcal{F} is quasinormal of order q and no subsequence of (f_k) is normal at the points a_j , the sequence (f_k) is normal in $D \setminus \{a_1, \dots, a_q\}$. If a subsequence of (f_k) would tend to a finite limit function in $D \setminus \{a_1, \dots, a_q\}$, then, by the maximum principle, this subsequence would be locally bounded and thus normal in D . Thus $f_k \rightarrow \infty$ in $D \setminus \{a_1, \dots, a_q\}$.

The following result [19, Theorem A.6] is a simple consequence of Lemma 2.5.

Lemma 2.6. *Let \mathcal{F} be a family of functions holomorphic in a domain D . Suppose that \mathcal{F} is quasinormal of order q . If the functions in \mathcal{F} are bounded at $q + 1$ points of D , then \mathcal{F} is normal.*

3. PROOF OF THEOREM 1.1

First we prove that \mathcal{F} is quasinormal of order at most 5. The argument in this part of the proof is essentially the same as in [3]. We refer to this paper for further details.

Assuming that \mathcal{F} is not quasinormal of order at most 5 we find six points $a_1, \dots, a_6 \in D$ and a sequence (f_k) in \mathcal{F} such that no subsequence of (f_k) is normal in a neighborhood of any a_j . We choose $\varepsilon < \min_{i \neq j} |a_i - a_j|$ and $\delta < \varepsilon/4K$. For fixed k we consider the digraph $G = (V, E)$ whose vertices are the a_j and whose edges are all pairs (a_i, a_j) for which f_k has a simple island over $D(a_j, \varepsilon)$ which is contained in $D(a_i, \delta)$. It follows from Lemma 2.1 that if k is large enough, then the outdegree of each vertex is at least 4. Lemma 2.4 now shows that if $n \geq 2$, then G contains a primitive closed walk $(a_{i_0}, a_{i_1}, \dots, a_{i_n})$ of length n . Thus $D(a_{i_{n-1}}, \delta)$ contains a simple island U_{n-1} over $D(a_{i_n}, \varepsilon)$. Next $D(a_{i_{n-2}}, \delta)$ contains a simple island over $D(a_{i_{n-1}}, \varepsilon)$ and thus in particular a simple island U_{n-2} over U_{n-1} . Inductively we find simple islands $U_j \subset D(a_{i_j}, \delta)$ over U_{j+1} , for $j = 0, 1, \dots, n - 2$. We deduce that f_k^n maps U_0 bijectively onto $D(a_{i_n}, \varepsilon)$. Since $U_0 \subset D(a_{i_0}, \delta) = D(a_{i_n}, \delta)$ we deduce from Lemma 2.3 that f_k^n has a fixed point $\xi \in U_0 \subset D(a_{i_0}, \delta)$ with $|(f_k^n)'(\xi)| \geq \varepsilon/4\delta > K$. Since the walk $(a_{i_0}, a_{i_1}, \dots, a_{i_n})$ is primitive, this fixed point ξ of f_k^n is in fact a periodic point of period n of f_k . This contradicts the assumption that $f_k \in \mathcal{F}$.

Hence \mathcal{F} is quasinormal of order q for some $q \leq 5$. We have to show that $q = 1$ and thus assume that $q \geq 2$. As before there exist q points $a_1, \dots, a_q \in D$ and a sequence (f_k) in \mathcal{F} such that no subsequence of (f_k) is normal in a neighborhood of any a_j . From Lemma 2.5 we deduce that $f_k \rightarrow \infty$ in $D \setminus \{a_1, \dots, a_q\}$.

Again we choose $\varepsilon < \min_{i \neq j} |a_i - a_j|$ and $\delta < \varepsilon/4K$. We also choose $R > \max_j |a_j| + \varepsilon$ and find that if k is sufficiently large, then $|f_k(z)| > R$ for $|z - a_j| = \delta$ and $j = 1, \dots, q$. On the other hand, since no subsequence of (f_k) is normal at a_j , we have $|f_k(z)| < R$ for some $z \in D(a_j, \delta)$ if k is large. Thus $f_k^{-1}(D(0, R))$ has a component $U_j \subset D(a_j, \delta)$ for $j = 1, \dots, q$, provided that k is large enough. Clearly $(f_k|_{U_j}, U_j, D(0, R))$ is a polynomial-like map.

By Lemma 2.2 there exists two simple islands over $D(a_1, \varepsilon)$ or $D(a_2, \varepsilon)$ contained in U_1 , and two further simple islands over $D(a_1, \varepsilon)$ or $D(a_2, \varepsilon)$ contained in U_2 . Without loss of generality we may assume that two of these four simple islands are over $D(a_1, \varepsilon)$; that is, we have two simple islands V_1, W_1 over $D(a_1, \varepsilon)$ which are contained in $U_1 \cup U_2$. Using Lemma 2.2 we now find for $j = 2, 3, \dots, n-1$ domains $V_j, W_j \subset U_2$ which are simple islands over V_{j-1} or W_{j-1} , and finally domains $V_n, W_n \subset U_1$ which are simple islands over V_{n-1} or W_{n-1} . Then f^n maps V_n bijectively onto $D(a_1, \varepsilon)$. Lemma 2.3 shows that f^n has a fixed point $\xi \in V_n$ with $|(f^n)'(\xi)| > K$. Moreover, $f^j(\xi) \in V_{n-j} \cup W_{n-j} \subset U_2 \subset D(a_2, \delta)$ for $j = 1, 2, \dots, n-2$, and this implies that ξ is in fact a periodic point of period n provided that $n \geq 3$.

We now consider the case that $n = 2$. If V_1, W_1 are both contained in U_2 , then the above argument also works for $n = 2$; that is, the fixed point $\xi \in V_2 \subset U_1$ of f^2 constructed above satisfies $f(\xi) \in V_1 \cup W_1 \subset U_2$ and thus is not a fixed point of f , but a periodic point of period 2. Next we consider the case that V_1, W_1 are both contained in U_1 . Since V_1 is a simple island over $D(a_1, \varepsilon)$ and $W_1 \subset U_1 \subset D(a_1, \varepsilon)$, we see that V_1 contains a simple island X over W_1 . Since f^2 maps X bijectively onto $D(a_1, \varepsilon)$, Lemma 2.2 again yields a fixed point $\xi \in X$ of f^2 with $|(f^2)'(\xi)| > K$. Since $f(\xi) \in W_1$ and $X \cap W_1 \subset V_1 \cap W_1 = \emptyset$, we have $f(\xi) \neq \xi$ so that ξ has period 2. Finally we consider the case that both U_1 and U_2 contain only one island over $D(a_1, \varepsilon)$, say $V_1 \subset U_1$ and $W_1 \subset U_2$. Then $U_1 \cup U_2$ also contains two simple islands X_1, Y_1 over $D(a_2, \varepsilon)$. We may assume that one of them is contained in U_1 and one is contained in U_2 , say $X_1 \subset U_1$ and $Y_1 \subset U_2$, since otherwise we are in one of the situations already considered. Now X_1 contains a simple island Z over W_1 and thus f^2 maps Z bijectively onto $D(a_1, \varepsilon)$. As before we deduce from Lemma 2.2 that f has a periodic point $\xi \in Z$ of period 2 with $|(f^2)'(\xi)| > K$.

We have thus obtained a contradiction to the assumption that $q \geq 2$ in all cases.

4. PROOF OF THEOREM 1.2

The basic idea is essentially the same as in [3, §4]. Let (c_k) be a sequence tending to ∞ such that $(f(c_k))$ is bounded. Define $f_k : \mathbb{C} \rightarrow \mathbb{C}$ by $f_k(z) = f(c_k z)/c_k$. Then $f_k(0) \rightarrow 0$ and $f_k(1) \rightarrow 0$.

Suppose now that the conclusion does not hold. Then there exists $K > 1$ such that $|(f^n)'(\xi)| \leq K$ for all periodic points ξ of period n of f . We note that if ξ is a periodic point of f_k of period n , then $c_k \xi$ is a periodic point of f of the same period, and $(f_k^n)'(\xi) = (f^n)'(c_k \xi)$. Thus the sequence (f_k) is quasnormal of order 1 by Theorem 1.1. Since (f_k) is bounded at 0 and 1, Lemma 2.6 yields that (f_k) is normal.

On the other hand, it is not difficult to see that (f_k) is not normal at 0. For example, this follows since $M(r, f) := \max_{|z|=r} |f(z)| \geq r^2$ for sufficiently large r so that

$$M\left(\frac{1}{\sqrt{|c_k|}}, f_k\right) = \frac{M(\sqrt{|c_k|}, f)}{|c_k|} \geq 1$$

for large k , while $f_k(0) \rightarrow 0$. Thus no subsequence of (f_k) can converge in a neighborhood of 0.

5. PROOF OF THEOREM 1.3

The proof of Theorem 1.3 is based on the following lemma due to X. Pang and L. Zalcman [17, Lemma 2].

Lemma 5.1. *Let \mathcal{F} be a family of functions meromorphic on the unit disc, all of whose zeros have multiplicity at least k , and suppose that there exists $A \geq 1$ such that $|g^{(k)}(\xi)| \leq A$ whenever $g(\xi) = 0$, $g \in \mathcal{F}$. Then if \mathcal{F} is not normal there exist, for each $0 \leq \alpha \leq k$, a number $r \in (0, 1)$, points $z_j \in D(0, r)$, functions $g_j \in \mathcal{F}$ and positive numbers ρ_j tending to zero such that*

$$\frac{g_j(z_j + \rho_j z)}{\rho_j^\alpha} \rightarrow G(z)$$

locally uniformly, where G is a nonconstant meromorphic function on \mathbb{C} such that the spherical derivative $G^\#$ of G satisfies $G^\#(z) \leq G^\#(0) = kA + 1$ for all $z \in \mathbb{C}$.

We shall only need the case $k = 1$ of Lemma 5.1. This special case can also be found in [16, Lemma 2].

We also require the following result of Essén and Wu [11, Theorem 4].

Lemma 5.2. *Let f be a polynomial of degree at least 2 and let $n \geq 2$. Then f^n has a repelling fixed point.*

Proof of Theorem 1.3. We denote by $\mathcal{F}(D, n, K)$ the family of all functions f holomorphic in D such that $|(f^n)'(\xi)| \leq K$ whenever $f^n(\xi) = \xi$. Note that this implies that $|f'(\xi)| \leq \sqrt[n]{K}$ whenever $f(\xi) = \xi$. We fix $n \geq 2$ and suppose that there does not exist $K > 1$ such $\mathcal{F}(D, n, K)$ is normal, in order to seek a contradiction. We may assume that D is the unit disk.

For $m \in \mathbb{N}$ we choose a non-normal sequence (f_j) in $\mathcal{F}(D, n, 1 + 1/m)$. With $g_j(z) := f_j(z) - z$ we find that if $g_j(\xi) = 0$, then $f_j(\xi) = \xi$ and thus

$$|g'_j(\xi)| \leq |f'_j(\xi)| + 1 \leq \sqrt[n]{1 + \frac{1}{m}} + 1 \leq 3.$$

Clearly, the sequence (g_j) is also not normal. Applying Lemma 5.1 for $\alpha = k = 1$ and $A = 3$ we may assume, passing to a subsequence if necessary, that there exist $z_j \in D$ and $\rho_j > 0$ such that $g_j(z_j + \rho_j z)/\rho_j \rightarrow G_m(z)$ for some entire function G_m satisfying $G_m^\#(z) \leq G_m^\#(0) = 4$ for all $z \in \mathbb{C}$. With $L_j(z) = z_j + \rho_j z$ we find that

$$h_j(z) := L_j^{-1}(f_j(L_j(z))) = \frac{f_j(z_j + \rho_j z) - z_j}{\rho_j} = \frac{g_j(z_j + \rho_j z)}{\rho_j} + z \rightarrow G_m(z) + z.$$

With $F_m(z) := G_m(z) + z$ we thus have $h_j(z) \rightarrow F_m(z)$ as $j \rightarrow \infty$. It follows that $h_j^n(z) \rightarrow F_m^n(z)$. The assumption that $f_j \in \mathcal{F}(D, n, 1 + 1/m)$ implies that $h_j \in \mathcal{F}(L_j^{-1}(D), n, 1 + 1/m)$; that is, $|(h_j^n)'(\xi)| \leq 1 + 1/m$ whenever $h_j^n(\xi) = \xi$. We deduce that $F_m \in \mathcal{F}(\mathbb{C}, n, 1 + 1/m)$. Since $G_m^\#(z) \leq G_m^\#(0) = 4$ for all $z \in \mathbb{C}$, the sequence (G_m) is normal. Hence (F_m) is normal and we may assume without loss of generality that $F_m \rightarrow F$ for some non-constant entire function F . Since $F_m \in \mathcal{F}(\mathbb{C}, n, 1 + 1/m)$ we conclude that $F \in \mathcal{F}(\mathbb{C}, n, 1)$; that is, F is an entire function such that F^n has no repelling fixed point. We deduce from Lemma 5.2 and Theorem D (or Theorem 1.2) that F is a polynomial of degree 1 at most. Next we note that $|F'_m(0)| \geq |G'_m(0)| - 1 \geq G_m^\#(0) - 1 = 3$ and thus $|F'(0)| \geq 3$. Hence F has the form $F(z) = az + b$ where $|a| \geq 3$. With $\xi := b/(1 - a)$ we obtain $F(\xi) = \xi$ and $|F'(\xi)| = |a| \geq 3$, contradicting $F \in \mathcal{F}(\mathbb{C}, n, 1)$.

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