FIXED POINTS OF COMPOSITE MEROMORPHIC FUNCTIONS AND NORMAL FAMILIES

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Abstract. We show that there exists a function f meromorphic in the plane $\mathbb{C}$ such that the family of all functions g holomorphic in the unit disk $\mathbb{D}$ for which $f \circ g$ has no fixed point in $\mathbb{D}$ is not normal. This answers a question of Hinchliffe who had shown that this family is normal if $\widehat{\mathbb{C}} \backslash f(\mathbb{C})$ does not consist of exactly one point in $\mathbb{D}$. We also investigate the normality of the family of all holomorphic functions g such that $f(g(z)) \neq h(z)$ for some non-constant meromorphic function h.

1. Introduction

A heuristic principle in complex function theory attributed to Bloch says that if a certain property forces an entire function to be constant, then the family of all functions holomorphic in some domain which have this property is likely to be normal; see [14, 15] for a thorough discussion of Bloch's principle. More generally, one might expect that a property satisfied by "very few" entire functions makes a family of holomorphic functions normal. Here we are concerned with normal family analogues of the following result.

Theorem A. *Let f be meromorphic in $\mathbb{C}$ and let g be entire, both functions being neither constant nor linear, and at least one of them transcendental. Then the composite function $f \circ g$ has infinitely many fixed points.*

For a proof we refer to [1, 2] in the case where f and g are both transcendental, and to [11, Theorem 1] or [12, p. 200] for the case that f is rational. We mention that the case where f is a polynomial was dealt with earlier in [13]. The case that g is a polynomial follows from the observation [7, Lemma 3] that $f \circ g$ has infinitely many fixed points if and only if $g \circ f$ does. This case is also treated in [12, Theorem 4]. We note that if f is rational of degree at least 3, then the above conclusion holds for transcendental meromorphic g as well, but if f has degree 2, then this is not the case [9]. We mention that Theorem A answered a question by Gross; see [4, p. 542] and [8, Problem 5].

In order to obtain normal family analogues one may fix a function f meromorphic in $\mathbb{C}$ and consider for a domain $D \subset \mathbb{C}$ the family $\mathcal{G}_f$ of all holomorphic functions $g : D \to \mathbb{C}$ such that $f \circ g$ has no fixed point in D. Fang and Yuan [5, Theorem 1] showed that if f is polynomial of degree at least 2, then $\mathcal{G}_f$ is indeed normal.

For the case that f is transcendental the following result was proved by Hinchliffe [10].

Theorem B. *Let $f : \mathbb{C} \to \widehat{\mathbb{C}} := \mathbb{C} \cup \{\infty\}$ be transcendental and meromorphic, and let $\mathcal{G}_f$ be as above. If $\mathcal{G}_f$ is not normal at $\alpha \in D$, then $\widehat{\mathbb{C}} \setminus f(\mathbb{C}) = \{\alpha\}$.*

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In particular it follows that $\mathcal{G}_f$ can fail to be normal at at most one point of D, and that $\mathcal{G}_f$ is normal in D if f is entire or if $\widehat{\mathbb{C}} \setminus f(\mathbb{C})$ consists of two points.

We show that if $\widehat{\mathbb{C}} \setminus f(\mathbb{C}) = \{\alpha\}$, then $\mathcal{G}_f$ may not be not normal at α, thereby answering a question by Hinchliffe. We restrict to the case that $\alpha = 0$ and $D = \mathbb{D} := \{z \in \mathbb{C} : |z| < 1\}$.

Theorem 1. *There exists a transcendental meromorphic function $f : \mathbb{C} \to \widehat{\mathbb{C}}$ and a sequence (g_k) of entire function which is not normal at 0 and has the property that $f(g_k(z)) \neq z$ for all $k \in \mathbb{N}$ and all $z \in \mathbb{D}$.*

Theorem 1 follows from the following result.

Theorem 2. *There exists a transcendental entire function h and sequences (a_k) and (b_k) of positive real numbers such that $a_k \to \infty$, $b_k \to \infty$, $b_k/a_k \to \infty$ and $h(z) \neq b_k/(z - a_k)$ for $|z| < a_k + b_k$ and all $k \in \mathbb{N}$.*

In fact, with $g_k(z) := a_k + b_k z$ we have $h(g_k(z)) \neq b_k/(g_k(z) - a_k) = 1/z$ as long as $|g_k(z)| < a_k + b_k$, which is satisfied in particular for $z \in \mathbb{D}$. With $f := 1/h$ we thus have $f(g_k(z)) \neq z$ for $z \in \mathbb{D}$. Also, $g_k(0) = a_k \to \infty$ while $g_k(-a_k/b_k) = 0$. Since $-a_k/b_k \to 0$ this implies that (g_k) is not normal at 0.

Next we note in several of the papers cited after Theorem A not only fixed points of $f \circ g$ were considered, but more generally zeros of $f \circ g - h$ for suitable functions h. Here we only mention the result [3] that if f is transcendental and meromorphic in the plane and g is transcendental entire, then $f \circ g - h$ has infinitely many zeros for any non-constant rational function h. We show that Hinchliffe's result admits a similar generalization.

Theorem 3. *Let $f : \mathbb{C} \to \widehat{\mathbb{C}}$ be meromorphic and neither constant nor linear, let $D \subset \mathbb{C}$ be a domain, let $h : D \to \widehat{\mathbb{C}}$ be meromorphic and non-constant and let $\mathcal{G}$ be the family of all holomorphic functions $g : D \to \mathbb{C}$ such that $f(g(z)) \neq h(z)$ for all $z \in D$. If $\mathcal{G}$ is not normal at $\alpha \in D$, then $\widehat{\mathbb{C}} \setminus f(\mathbb{C}) = \{h(\alpha)\}$.*

Theorem 3 also applies to rational f, but in this case we have the following stronger result.

Theorem 4. *Let $f : \mathbb{C} \to \widehat{\mathbb{C}}$ be a rational function of degree at least 2, let $D \subset \mathbb{C}$ be a domain, let $h : D \to \widehat{\mathbb{C}}$ be meromorphic and non-constant and let $\mathcal{G}$ be the family of all holomorphic functions $g : D \to \mathbb{C}$ such that $f(g(z)) \neq h(z)$ for all $z \in D$. Then $\mathcal{G}$ is normal.*

This result generalizes a result of Fang and Yuan [6, Theorem 4] dealing with the case that f is a polynomial.

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2. Proof of Theorem 2

We shall construct the sequences (a_n) and (b_n) and a further sequence (h_n) of entire functions by recursion such that the following properties hold:

(i) if $m \geq 2$, then $a_m > a_{m-1} + 1$, $b_m > b_{m-1} + 1$ and $b_m > a_m^2$;
(ii) if $1 \leq k \leq m$ and $|z| \leq a_k + b_k$, then $h_m(z) \neq b_k/(z - a_k)$;
(iii) if $m \geq 2$, then $|h_m(z) - h_{m-1}(z)| \leq 2^{-m}$ for $|z| \leq a_{m-1} + b_{m-1}$.

It then follows from (i) that $r_n := a_n + b_n \to \infty$ as $n \to \infty$ and thus we can deduce from (iii) that (h_n) converges locally uniformly to some entire function h. Hurwitz's theorem and (ii) imply that $h(z) \neq b_k/(z - a_k)$ for $k \in \mathbb{N}$ and $|z| < a_k + b_k$.

To construct the sequences (a_n), (b_n) and (h_n) we take $a_1 = b_1 = 1$ and

$$h_1(z) = \frac{1 - e^{z-1}}{z - 1}.$$

Then (ii) clearly holds for $m = 1$.

We now assume that $n \geq 2$ and that $a_1, \ldots, a_{n-1}, b_1, \ldots, b_{n-1}, h_1, \ldots, h_{n-1}$ have been defined such that (i), (ii), (iii) are satisfied for $m \leq n - 1$. In particular, $h_{n-1}(z) \neq b_k/(z - a_k)$ for $|z| \leq r_k$ if $1 \leq k \leq n - 1$. Thus

$$\delta_k := \min_{|z| \leq r_k} \left| h_{n-1}(z) - \frac{b_k}{z - a_k} \right| > 0$$

for $1 \leq k \leq n - 1$. We define $\delta := \min\{\delta_1, \ldots, \delta_{n-1}, 2^{-n}\}$. We shall show that we can choose a_n, b_n and h_n such that that (i) is satisfied for $m = n$ and such that

$$|h_n(z) - h_{n-1}(z)| < \delta \quad \text{for } |z| \leq r_{n-1} \tag{1}$$

and

$$h_n(z) \neq \frac{b_n}{z - a_n} \quad \text{for } |z| \leq r_n. \tag{2}$$

It follows from these conditions that (ii) and (iii) are also satisfied for $m = n$. In fact, (iii) follows immediately from (1) since $\delta \leq 2^{-n}$. And for $1 \leq k \leq n - 1$ we deduce from (1) that

$$\left| h_n(z) - \frac{b_k}{z - a_k} \right| \geq \left| h_{n-1}(z) - \frac{b_k}{z - a_k} \right| - |h_n(z) - h_{n-1}(z)| > \delta_k - \delta \geq 0$$

for $|z| \leq r_k$, so that $h_n(z) \neq b_k/(z - a_k)$ for $|z| \leq r_k$ if $1 \leq k \leq n - 1$. But for $k = n$ this holds by (2) so that (ii) follows for $m = n$.

It remains to show that we can choose a_n and b_n according to (i) and then find h_n satisfying (1) and (2). In order to this we can take any $a_n > r_{n-1} > a_{n-1} + 1$ and choose b_n satisfying (i) with

$$b_n > 2 \max_{|z| = r_{n-1}} |(z - a_n) h_{n-1}(z)|.$$

Then

$$v(z) := \log \left(1 - \frac{(z - a_n) h_{n-1}(z)}{b_n} \right)$$

is holomorphic for $|z| \leq r_{n-1}$. By the definition of v we have

$$h_{n-1}(z) = \frac{b_n}{z - a_n} \left(1 - e^{v(z)} \right). \tag{3}$$

Next we note that there exists a sequence (v_d) of polynomials satisfying $v_d(a_n) = v_d'(a_n) = 0$ such that $v_d(z) \to v(z)$ as $d \to \infty$, uniformly for $|z| \leq r_{n-1}$. For example, we may choose

$$v_d(z) := T_d(z) - T_d(a_n) \left(\frac{z}{a_n} \right)^{\ell_d} - \left(T_d'(a_n) - T_d(a_n) \frac{\ell_d}{a_n} \right) (z - a_n) \left(\frac{z}{a_n} \right)^{m_d},$$

where T_d is the Taylor polynomial of degree d of v and ℓ_d and m_d are suitably chosen large integers (with ℓ_d chosen first and m_d also depending on ℓ_d). Then

$$h_n(z) := \frac{b_n}{z-a_n}\left(1-e^{v_d(z)}\right).$$

is entire, with $h_n(a_n)=0$. If d is large enough, then (1) is satisfied because of (3). Moreover,

$$h_n(z) \neq \frac{b_n}{z-a_n}$$

for all $z\in\mathbb{C}$, and thus in particular for $|z|\leq r_n$, so that (2) is also satisfied.

It remains to prove that h is transcendental. But this follows since

$$|h(0)| \geq |h_1(0)| - \sum_{m=2}^{\infty} |h_m(0)-h_{m-1}(0)| \geq 1-\frac{1}{e}-\sum_{m=2}^{\infty}\frac{1}{2^m} = \frac{1}{2}-\frac{1}{e} > 0$$

and

$$|h(a_n)| \leq |h_n(a_n)| + \sum_{m=n+1}^{\infty} |h_m(a_n)-h_{m-1}(a_n)| \leq \sum_{m=n+1}^{\infty}\frac{1}{2^m} = \frac{1}{2^n}.$$

3. Proof of Theorem 3

Theorem 3 can be proved by a modification of Hinchliffe's argument. For the first part of the proof (Lemma 1 below), however, we use a somewhat different argument. The remaining parts (Lemmas 3–6) are similar to Hinchliffe's argument.

Lemma 1. *Let $\mathcal{G}$ be as in Theorem 3. If there exists $\beta_1,\beta_2\in\mathbb{C}$ with $\beta_1\neq\beta_2$ and $f(\beta_1)=f(\beta_2)=h(\alpha)$, then $\mathcal{G}$ is normal at α.*

Our proof of Lemma 1 is based on the following lemma due to Zalcman [14], which was also used by Fang and Yuan [5].

Lemma 2. *Let $\mathcal{G}$ be a family of functions meromorphic in a domain $D\subset\mathbb{C}$ Suppose that $\mathcal{G}$ is not normal at $z_0\in D$. Then there exist a sequence (g_k) in $\mathcal{G}$, a sequence (z_k) in D, a sequence (ρ_k) of positive real numbers and a non-constant function g which is meromorphic in $\mathbb{C}$ such that $z_k\to z_0$, $\rho_k\to 0$ and $g_k(z_k+\rho_k z)\to g(z)$ locally uniformly in $\mathbb{C}$.*

Proof of Lemma 1. Without loss of generality we may assume that $h(\alpha)\neq\infty$ because otherwise we replace h and f by $1/h$ and $1/f$. Suppose $\mathcal{G}$ is not normal at α. By Lemma 2 there exist a sequence (g_k) in $\mathcal{G}$, a sequence (α_k) in D, a sequence (ρ_k) of positive real numbers and a non-constant function g which is meromorphic in $\mathbb{C}$ such that $\alpha_k\to\alpha$, $\rho_k\to 0$ and $g_k(\alpha_k+\rho_k z)\to g(z)$ locally uniformly in $\mathbb{C}$. By Picard's theorem g takes one of the values β_1 and β_2, say $g(\gamma)=\beta_1$. Now

$$F_k(z) := f(g_k(\alpha_k+\rho_k z)) - h(\alpha_k+\rho_k z) \to F(z) := f(g(z))-h(\alpha).$$

Since $F(\gamma)=f(\beta_1)-h(\alpha)=0$ and F is non-constant we deduce that for sufficiently large k there exists $\gamma_k\in\mathbb{C}$ satisfying $F_k(\gamma_k)=0$ and $\gamma_k\to\gamma$ as $k\to\infty$. For $\zeta_k:=\alpha_k+\rho_k\gamma_k$ we deduce that $0=F_k(\gamma_k)=f(g(\zeta_k)-h(\zeta_k)$. Thus $f(g(\zeta_k)=h(\zeta_k)$, contradicting the hypothesis. ☐

As already mentioned, the following lemmas and their proofs are similar to the arguments in [10].

We denote by $D(\alpha,r)$ the open disk of radius r around α; that is, $D(\alpha,r):=\{z\in\mathbb{C}:|z-\alpha|<r\}$.

Lemma 3. *Let $\mathcal{G}$ be a family of functions holomorphic in $D(\alpha, r)$. Suppose that $\mathcal{G}$ is normal in $D(\alpha, r)\backslash\{\alpha\}$, but not in α. Then there exist a sequence (g_k) in $\mathcal{G}$, a sequence (α_k) in $D(\alpha, r)$ and a positive real number M such that $|g_k(\alpha_k)| \leq M$, $\alpha_k \to \alpha$ and $g_k \to \infty$ locally uniformly in $D(\alpha, r)\backslash\{\alpha\}$.*

Proof. It follows from the hypothesis that there exists a sequence (g_k) in $\mathcal{G}$ which converges locally uniformly in $D(\alpha, r)\backslash\{\alpha\}$, but does not have a subsequence which converges locally uniformly in $D(\alpha, r)$.

Now $g_k \to g$ in $D(\alpha, r)\backslash\{\alpha\}$ for some function g. If $g \not\equiv \infty$, then g is holomorphic and thus there exists $K > 0$ such that $|g(z)| \leq K$ for $|z - \alpha| = r/2$. For sufficiently large k we have $|g_k(z)| \leq K + 1$ for $|z - \alpha| = r/2$, and thus $|g_k(z)| \leq K + 1$ for $|z - \alpha| < r/2$ by the maximum principle. It follows that (g_k) is normal in $D(\alpha, r/2)$ and thus has a subsequence which converges there, contradicting our assumption. Thus $g \equiv \infty$; that is, $g_k \to \infty$ in $D(\alpha, r)\backslash\{\alpha\}$.

On the other hand, since no subsequence of (g_k) converges locally uniformly to ∞ in $D(\alpha, r)$, there exists a sequence (α_k) in $D(\alpha, r)\backslash\{\alpha\}$ such that $\alpha_k \to \alpha$ and $g_k(\alpha_k)$ remains bounded, say $|g_k(\alpha_k)| \leq M$. □

Lemma 4. *Let $\mathcal{G}$ be as in Theorem 3. If $f^{-1}(h(\alpha)) = \{\beta\}$ for some $\beta \in \mathbb{C}$, then $\mathcal{G}$ is normal at α.*

Proof. Using Lemma 1 and the hypothesis that h is non-constant (which was not used in the proof of Lemma 1), we find that $\mathcal{G}$ is normal in some punctured neighborhood of α, say in $D(\alpha, r)\backslash\{\alpha\}$. We no suppose that $\mathcal{G}$ is not normal at α and choose (g_k), (α_k) and M according to Lemma 3.

We may again assume that $h(\alpha) \neq \infty$. We find $\rho, \varepsilon > 0$ such that $|f(z) - h(\alpha)| > \varepsilon$ for $|z - \beta| = \rho$. Next we choose δ with $0 < \delta < r$ such that $|h(z) - h(\alpha)| < \varepsilon$ for $|z - \alpha| < \delta$.

Since $g_k \to \infty$ on $D(\alpha, r)\backslash\{\alpha\}$ we see that if k is sufficiently large, then

$$|(g_k(z) - w) - (g_k(z) - g_k(\alpha_k))| = |w - g_k(\alpha_k)| \leq |\beta| + \rho + M < |g_k(z) - g_k(\alpha_k)|$$

for $|z - \alpha| = \delta$ and $w \in D(\beta, \rho)$. For large k we also have $|\alpha_k - \alpha| < \delta$, and thus we deduce from Rouché's theorem that g takes the value w in $D(\alpha, \delta)$; that is, we have $g_k(D(\alpha, \delta)) \supset D(\beta, \rho)$ for large k. Since also $g_k(\partial D(\alpha, \delta)) \cap D(\beta, \rho) = \emptyset$ for large k, we find a component U of $g_k^{-1}(D(\beta, \rho))$ contained in $D(\alpha, \delta)$ for such k. Moreover, U is a a Jordan domain and $g_k : U \to D(\beta, \rho)$ is a proper map.

For $z \in \partial U$ we then have $g_k(z) \in \partial D(\beta, \rho)$ and thus $|f(g_k(z)) - h(\alpha)| > \varepsilon$. Hence

$$|(f(g_k(z)) - h(\alpha)) - (f(g_k(z)) - h(z))| = |h(z) - h(\alpha)| < \varepsilon < |f(g_k(z)) - h(\alpha)|$$

for $z \in \partial U$. Now g_k in particular takes the value β in U, say $g_k(\gamma_k) = \beta$ with $\gamma_k \in U$. Hence $f(g_k(\gamma_k)) - h(\alpha) = 0$, and thus Rouché's theorem now shows that $f \circ g_k - h$ has a zero in U, a contradiction. □

Lemma 5. *Let $\mathcal{G}$ be as in Theorem 3. If $\widehat{\mathbb{C}} \setminus f(\mathbb{C})$ consists of two points, then $\mathcal{G}$ is normal.*

For the proof of Lemma 5 we shall use the following result.

Lemma 6. *Let $\mathcal{F}$ be a family of functions meromorphic in $D(\alpha, r)$ for some $\alpha \in \mathbb{C}$ and $r > 0$. Suppose that there exist distinct $\beta, \gamma \in \widehat{\mathbb{C}}$ such that $f(z) \neq \beta, \gamma$ for all $z \in D$. If $\mathcal{F}$ is normal in $D(\alpha, r)\backslash\{\alpha\}$, then $\mathcal{F}$ is normal in $D(\alpha, r)$.*

Proof. We may assume that $\beta = 0$ and $\gamma = \infty$ because otherwise we can consider $\{M \circ f\}_{f \in \mathcal{F}}$ with a suitable Möbius transformation M instead of $\mathcal{F}$.

Suppose that $\mathcal{F}$ is normal in $D(\alpha, r)\backslash\{\alpha\}$, but not normal at α. Choose (g_k), (α_k) and M according to Lemma 3. Now $1/g_k$ is holomorphic in $D(\alpha, r)$ and tends to 0 locally uniformly in $D(\alpha, r)\backslash\{\alpha\}$. It follows from the maximum principle that $1/g_k \to 0$ in $D(\alpha, r)\backslash\{\alpha\}$, contradicting $|g_k(\alpha_k)| \leq M$. □

Proof of Lemma 5. We may assume that $\widehat{\mathbb{C}} \setminus f(\mathbb{C}) = \{0, \infty\}$, because otherwise we can consider $M \circ f$ and $M \circ h$ instead of f and h for a suitable Möbius transformation M. We consider the family $\mathcal{F}$ consisting of all functions F of the form $F(z) = f(g(z))/h(z)$ where $g \in \mathcal{G}$. It follows from the hypothesis that if $F \in \mathcal{F}$, then $F(z) \neq 1$ for all $z \in D$. Moreover, the only zeros and poles of the functions in $\mathcal{F}$ are the poles and zeros of h. It thus follows from Montel's theorem that $\mathcal{F}$ is normal in $D\backslash S$, where S denotes the set of zeros and poles of h. Since every $F \in \mathcal{F}$ satisfies $F(z) \neq 0, 1$ near zeros of h and $F(z) \neq 1, \infty$ near poles of h, we can deduce from Lemma 6 that $\mathcal{F}$ is in fact normal in D. It is not difficult to see that this implies that $\mathcal{G}$ is normal in D. □

In order to prove Theorem 3 we now only have to combine Lemmas 1, 4 and 5.

4. Proof of Theorem 4

Suppose that $\mathcal{G}$ is not normal at $\alpha \in D$. Then $\widehat{\mathbb{C}} \setminus f(\mathbb{C}) = \{h(\alpha)\}$ by Theorem 3. This implies that $f(\infty) = h(\alpha)$. Again we may assume that $h(\alpha) \neq \infty$. Then f has a pole $p \in \mathbb{C}$.

Lemma 1 shows that $\mathcal{G}$ is normal in $D(\alpha, r)\backslash\{\alpha\}$ for some $r > 0$. Let (g_k), (α_k) and M be as in Lemma 3.

Choose ρ with $0 < \rho < r$ such that $h(z) \neq h(\alpha)$ and $h(z) \neq \infty$ for $0 < |z-\alpha| \leq \rho$. Since no subsequence of (g_k) is normal in $D(\alpha, \rho)$, Lemma 6 shows that g_k takes the value p in $D(\alpha, \rho)$ for sufficiently large k. Thus $f(g_k)$ has a pole there. Now $f(g_k(z)) \to h(\alpha)$ for $|z - \alpha| = \rho$ as $k \to \infty$. For sufficiently large k and $|z - \alpha| = \rho$ we thus have

$$|(f(g_k(z)) - h(z)) - (h(\alpha) - h(z))| = |f(g_k(z)) - h(\alpha)| < |h(\alpha) - h(z)|.$$

Rouché's theorem yields that for the functions $f(g_k(z)) - h(z)$ and $h(\alpha) - h(z)$ the difference between the number of zeros and poles in $D(\alpha, \rho)$ is equal. Now $h(\alpha) - h(z)$ has a zero there, but no pole. And $f(g_k(z)) - h(z)$ has a pole there. Thus $f(g_k(z)) - h(z)$ has (counting multiplicity) at least two zeros there. This is a contradiction. □

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