PERIODIC FATOU COMPONENTS AND SINGULARITIES OF THE INVERSE FUNCTION

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Abstract. The periodic components of the Fatou set of a rational or entire function are closely connected to the singularities of the inverse function. This article is a survey of classical as well as more recent results concerning this relation.

1. Introduction

1.1. BASIC DEFINITIONS

Throughout this paper, let f be a rational function of degree at least two or a transcendental entire function. We denote by $D(f)$ the domain of definition of f so that $D(f) = \mathbb{C}$ if f is entire transcendental while $D(f) = \widehat{\mathbb{C}} := \mathbb{C} \cup \{\infty\}$ if f is rational. The basic objects studied in complex dynamics are the Fatou set $F(f)$ of f, which is defined to be the set of all points in $D(f)$ where the iterates of f form a normal family, and the Julia set $J(f)$, which is the complement of $F(f)$ with respect to $D(f)$. Thus $J(f) = \mathbb{C} \backslash F(f)$ for entire transcendental f and $J(f) = \widehat{\mathbb{C}} \backslash F(f)$ for rational f.

We denote by f^n the n–th iterate of f, with $f^0 = \mathrm{id}_{D(f)}$. We say that $z_0 \in D(f)$ is a *preperiodic point* of f if there exist $p > q \geq 0$ such that $f^p(z_0) = f^q(z_0)$. In the special case that $q = 0$ so that $f^p(z_0) = z_0$ for some $p \geq 1$ we say that z_0 is a *periodic point* of f. The smallest p with this property is called the *period* of z_0. For a periodic point $z_0 \in \mathbb{C}$ of period p we call $(f^p)'(z_0)$ the *multiplier* of z_0. If $z_0 = \infty$, which can happen only for rational function f of course, this has to be modified: in this case, the multiplier is defined to be $(g^p)'(0)$ where $g(z) := 1/f(1/z)$. A periodic point is called *attracting*, *indifferent*, or *repelling* depending on whether the modulus of its multiplier is less than, equal to, or greater than 1. Periodic points of multiplier 0 are called *superattracting*. The multiplier of an indifferent periodic point is of the form $e^{2\pi i\alpha}$ where $0 \leq \alpha < 1$. We say that z_0 is *rationally indifferent* if α is rational and *irrationally indifferent* otherwise. Finally, a periodic point of period 1 is called a *fixed point*.

A basic result in complex dynamics says that the Julia set is the closure of the set of repelling periodic points. Among other basic properties of the Fatou and Julia set we mention here only that both sets are completely invariant. Here, by definition, a subset S of $D(f)$ is called *completely invariant* if $f(z) \in S$ if and only if $z \in S$. For a an introduction to complex dynamics we refer to the textbooks [9, 21, 26, 43, 56] for rational functions. The case of transcendental entire functions (but also that of rational functions) is treated in [10, 29, 45].

1.2. THE CLASSIFICATION OF PERIODIC FATOU COMPONENTS

A maximal domain of normality of the iterates of f, that is, a connected component of $F(f)$, is called a *Fatou component*. If U is a Fatou component, then $f^p(U)$ is contained in a Fatou component which we denote by U_p. A Fatou component U is called *preperiodic* if there exist $p > q \geq 0$ such that $U_p = U_q$. In particular, if this is the case for $q = 0$ (where $U_0 = U$) and some $p \geq 1$, then U is called *periodic*, and $\{U, U_1, \ldots, U_{p-1}\}$ is called a *periodic cycle* of Fatou components. Again, the smallest p with this property is called the *period* of U. In the case $p = 1$, that is, if $f(U) \subset U$, the Fatou component U is called *invariant*. A Fatou component which is not preperiodic is called a *wandering component* (or *wandering domain*).

For rational functions we have $f^p(U) = U_p$, but for transcendental functions it is possible that $f^p(U) \neq U_p$. However, $U_p \backslash f^p(U)$ contains at most one point; see [18, 22, 23, 37]

The behavior of the iterates in periodic components is well understood. Let U be a periodic Fatou component of period p. Then we have one of the following possibilities:

- U contains an attracting periodic point z_0 of period p. Then $f^{np}(z) \to z_0$ for $z \in U$ as $n \to \infty$. If z_0 is superattracting, then U is called a *Böttcher domain*. Otherwise U is called a *Schröder domain*.
- ∂U contains a fixed point z_0 of f^p and $f^{np}(z) \to z_0$ for $z \in U$ as $n \to \infty$. Then $(f^p)'(z_0) = 1$ if $z_0 \in \mathbb{C}$. (For $z_0 = \infty$ we have $(g^p)'(0) = 1$ where $g(z) := 1/f(1/z)$.) We call U a *Leau domain* at z_0.

- There exists an analytic homeomorphism $\phi : U \to \mathbb{D}$ where $\mathbb{D}$ is the unit disk such that $\phi(f^p(\phi^{-1}(z))) = e^{2\pi i\alpha}z$ for some $\alpha \in \mathbb{R}\backslash\mathbb{Q}$. In this case, U is called a *Siegel disk*.
- There exists an analytic homeomorphism $\phi : U \to A$ where A is an annulus, $A = \{z : 1 < |z| < r\}$, $r > 1$, such that $\phi(f^p(\phi^{-1}(z))) = e^{2\pi i\alpha}z$ for some $\alpha \in \mathbb{R}\backslash\mathbb{Q}$. In this case, U is called a *Herman ring*.
- $f^{np}(z) \to \infty \notin D(f)$ for $z \in U$ as $n \to \infty$. In this case, U is called a *Baker domain*.

Clearly, if f is rational, then Baker domains do not exist. Moreover, it is not difficult to see that entire functions do not have Herman rings.

The above classification of periodic Fatou components is due to Fatou and Cremer; see [10, p. 163] for a more detailed historic account with references. For a proof we refer to the textbooks mentioned in the introduction. Most of them deal only with the case that f is rational, but the case that f is entire transcendental requires only minor modifications. (This remark concerns also various other references made to these textbooks in this paper.)

If z_0 is an attracting periodic point, then z_0 is contained in a Schröder or Böttcher domain. Similarly, if z_0 is a periodic point of multiplier 1, then there is a Leau domain at z_0. More precisely, let $z_0 \in \mathbb{C}$ be a fixed point of multiplier 1; that is, $f(z_0) = z_0$ and $f'(z_0) = 1$. Then z_0 is a multiple pole of the function $h(z) := 1/(z - f(z))$, say of multiplicity $m + 1$ with $m \in \mathbb{N}$. (We also say that z_0 is a *multiple fixed point* of f of multiplicity $m + 1$.) We find that there are exactly m invariant Leau domains at z_0. There is of course an obvious modification of this result for the case that $z_0 = \infty$, as well as to the case that z_0 is not a fixed point of multiplier 1, but a rationally indifferent periodic point. The existence of Siegel disks and Herman rings was first shown by Siegel [54] and Herman [35]. The first example of an entire function with a Baker domain was already given by Fatou [32, Exemple I] who considered the function $f(z) = z + 1 + e^{-z}$ and noted that $\text{Re } f^n(z) \to \infty$ as $n \to \infty$ for $\text{Re } z > 0$ which implies that the right half-plane is contained in an invariant Baker domain. Examples of Baker domains of period greater than one were given in [44, 48]. The term "Baker domain" was introduced by Eremenko and Lyubich [29, 30].

1.3. SINGULARITIES OF THE INVERSE FUNCTION

An important role in complex dynamics is played by the singularities of the inverse function. Let $a \in D(f)$ and let $\gamma : [0, 1] \to D(f)$ be a curve in $D(f)$ with endpoint $\gamma(1) = a$. Let φ be a branch of f^{-1} defined in some neighborhood of $b := \gamma(0)$; that is, φ is meromorphic in some neighborhood V of b and $f(\varphi(z)) = z$ for all $z \in V$. Suppose that φ can be continued analytically along γ into the point $\gamma(t)$ for all $t \in [0,1)$, but not into the point $a = \gamma(1)$. Then (that is, if b, γ, φ as above exist) the point a is called a *singularity of the inverse function of* f.

One way this is possible is that $\varphi(\gamma(t)) \to z_0$ as $t \to 1$ for some $z_0 \in D(f)$ with the property that f is not locally univalent at z_0. Then $f(z_0) = a$, and we call z_0 a *critical point* and a a *critical value* of f. Note that if $z_0 \neq \infty$ and $f(z_0) \neq \infty$, then z_0 is a critical point if and only if $f'(z_0) = 0$. For rational functions f, critical values are the only singularities of f^{-1}. For transcendental entire f it is also possible that $\varphi(\gamma(t)) \to \infty \notin D(f)$ as $t \to 1$. Then a is called an *asymptotic value*.

We call the critical and asymptotic values of f also *singular values* and denote the set of all singular values of f by $\operatorname{sing}(f^{-1})$. We note that (see [1, Lemma 2])

$$\operatorname{sing}((f^p)^{-1}) = \bigcup_{n=0}^{p-1} f^n \left(\operatorname{sing}(f^{-1})\right). \tag{1}$$

The *postsingular set* $P(f)$ is defined by

$$P(f) := \overline{\bigcup_{n=0}^{\infty} f^n\left(\operatorname{sing}(f^{-1})\right)} = \overline{\bigcup_{n=0}^{\infty} \operatorname{sing}\left((f^n)^{-1}\right)}).$$

2. Relations between periodic Fatou components and singular values

By definition, a periodic cycle of Böttcher domains contains a superattracting fixed point and thus a critical point and a critical value of f. We shall discuss how the other types of Fatou components are related to singular values.

2.1. SCHRÖDER AND LEAU DOMAINS

The fundamental result in this case is the following theorem. A proof can be found in the textbooks mentioned in the introduction.

Theorem 1 *Let $\{U_0, U_1, \dots, U_{p-1}\}$ be a periodic cycle of Schröder or Leau domains of f. Then $U_j \cap \operatorname{sing}(f^{-1}) \neq \emptyset$ for some $j \in \{0, 1, \dots, p-1\}$. More precisely, there exists $j \in \{0, 1, \dots, p-1\}$ such that $U_j \cap \operatorname{sing}(f^{-1})$ contains a point which is not preperiodic.*

This result can already be found in the memoirs by Fatou [31, 32] and Julia [39] that founded the theory. More recently, it has been shown that under a suitable additional hypothesis a periodic cycle of Leau domains contains at least two singular values. In order to formulate this result, let

$z_0 \in \mathbb{C}$ be a fixed point of f. Then the function $h(z) := 1/(z - f(z))$ has a pole at z_0. The residue of h at z_0 is called the *residue fixed point index* and denoted by $\iota(f, z_0)$; see [43, §12] for a detailed discussion of this concept. The residue fixed point index is invariant under holomorphic changes of variables [43, Lemma 12.3]. This is used to define it for $z_0 = \infty$; that is, if ∞ is a fixed point of f and g is defined by $g(z) := 1/f(1/z)$, then $\iota(f, \infty) := \iota(g, 0)$. Of course, there is an obvious modification of the residue fixed point index to periodic points, but for simplicity we state the following result only for fixed points. It is easily transferred to the case of rationally indifferent periodic points using (1).

Theorem 2 *Let z_0 be a fixed point of f of multiplier* 1. *Let $m + 1$ be the multiplicity of z_0. If*

$$\operatorname{Re} \iota(f, z_0) \geq \frac{7}{20}m + \frac{1}{2}, \tag{2}$$

then one of the m Leau domains associated to z_0 contains at least two singular values.

This theorem follows from results obtained by Bergweiler [14] and Buff and Epstein [25], which extended previous work by Shishikura [52]. In fact, for rational f this theorem follows immediately from [14, Theorem 2]. Moreover, it was shown in [14, Theorem 2] that the conclusion of Theorem 2 holds for functions f meromorphic in the plane if

$$\operatorname{Re} \iota(f, z_0) > \frac{7}{20}m + \frac{1}{2}. \tag{3}$$

Here we refer to [10] for the basic definitions and results of complex dynamics in the setting of transcendental meromorphic functions. We note that we cannot replace (3) by (2) in this setting, as shown by the example $f(z) = \tan z$. Here $z_0 = 0$ is a fixed point of multiplicity 3, the two associated Leau domains being the upper and lower halfplane. As $\operatorname{sing}(\tan^{-1}) = \{i, -i\}$, we see that each of them contains only one singular value. A simple computation shows that $\iota(\tan, 0) = \frac{6}{5}$ so that we have equality in (2) for $m = 2$. Similarly, $f(z) = \tan^2 \sqrt{z}$ provides an example of a multiple fixed point of multiplicity 2 having only one singular value in its Leau domain, yielding equality in (2) for $m = 1$.

These examples show that the conclusion of Theorem 2 need not hold for transcendental meromorphic f if we have equality in (2). On the other hand, the methods of [25] show that if f is a transcendental meromorphic functions for which we have equality in (2) and the conclusion of Theorem 2 does not hold, then the Julia set of f must be a straight line or a line segment. But this is impossible for transcendental entire functions by a result of Töpfer [58, §3].

We note that estimates sharper than (2) are available if all singular values are critical values and if the multiplicities of the critical points are bounded. For example, if all singular values are critical values and if the critical points are simple, then (2) can be replaced by Re $\iota(f, z_0) > \frac{1}{4}m + \frac{1}{2}$; see [14, 25] for details.

2.2. SIEGEL DISKS AND HERMAN RINGS

The classical result in this case is the following theorem, whose proof can be found in the textbooks mentioned in the introduction.

Theorem 3 *Let* $\{U_0, U_1, \ldots, U_{p-1}\}$ *be a periodic cycle of Siegel disks or Herman rings. Then* $\partial U_j \subset P(f)$ *for all* $j \in \{0, 1, \ldots, p-1\}$.

A more recent result by Mañé ([42], see also [53]) says that the boundary of a Siegel disc of a rational function is contained in the ω-limit set of a recurrent critical point. The question when the boundary of a Siegel disk or Herman ring actually contains a point of $\text{sing}(f^{-1})$ is rather delicate; see [27, 33, 36, 46, 47, 50, 51, 59] for results in this direction.

2.3. BAKER DOMAINS

As rational functions do not have Baker domains, f will always denote an entire transcendental function in this section. The following result is due to Eremenko and Lyubich [30, Theorem 1].

Theorem 4 *If* $\text{sing}(f^{-1})$ *is bounded, then there is no Fatou component* U *such that* $f^n(z) \to \infty$ *for* $z \in U$ *as* $n \to \infty$. *In particular,* f *does not have Baker domains.*

The part concerning Baker domains was strengthened by Bargmann [7, Theorem 4] as follows.

Theorem 5 *If* f *has an invariant Baker domain, then there exist* $c > 1$ *and* $r_0 > 0$ *such that every annulus* $\{z : r < |z| < cr\}$ *with* $r \geq r_0$ *contains a singular value.*

In view of these results one might think that Baker domains always contain singular values. However, this is not the case. The first examples of entire functions f having a Baker domain U with $U \cap \text{sing}(f^{-1}) = \emptyset$ were given by Herman [36, p. 609] and Eremenko and Lyubich [28, Example 3]. As a specific example with this property we mention the function $f(z) = 2 - \log 2 + 2z - e^z$ which even has a Baker domain U such that $\text{dist}(P(f), U) > 0$, where $\text{dist}(\cdot, \cdot)$ denotes the Euclidean distance in the plane; see [12, Theorem 1] for this example. A detailed study of invariant Baker domains U satisfying $U \cap \text{sing}(f^{-1}) = \emptyset$ was given by Barański and Fagella [6]. Examples of periodic cycles of Baker domains of higher period which do not contain singular values were given by Rippon and Stallard [49, Theorem 4].

It was observed by Herman [36, p. 609] that Sullivan's method [57] to prove the non-existence of wandering domains for rational functions can also be applied to entire functions with Baker domains and leads to the following result.

Theorem 6 *If f has a periodic cycle of Baker domains which does not contain a singular value, then f has an infinite dimensional space of quasiconformal deformations.*

As specific examples where Theorem 6 applies we mention functions f of the form $f(z) = z + p(z)e^{q(z)}$ where p, q are polynomials, or functions f of the form $f(z) = z - g(z)/g'(z)$ where $g(z) = \int_0^z e^{q(t)}dt$ with a polynomial q. (Note that f arises from applying Newton's method for finding the zeros of g.) These functions have finite dimensional spaces of quasiconformal deformations, and thus every periodic cycle of Baker domains contains a singular value. We refer to [11, 55] for these examples, as well as to [20] for some other examples; see also [10, Theorem 14] for further discussion.

Another relation between Baker domains and singular values is given by the following result proved in [12, Theorem 3].

Theorem 7 *If U is an invariant Baker domain which does not contain a singular value, then there exists a sequence (p_n) such that $p_n \in P(f)$, $|p_n| \to \infty$, $|p_{n+1}/p_n| \to 1$, and $\text{dist}(p_n, U) = o(|p_n|)$ as $n \to \infty$.*

We note that the example $f(z) = 2 - \log 2 + 2z - e^z$ already mentioned shows that $\text{dist}(p_n, U) = o(|p_n|)$ and $|p_{n+1}/p_n| \to 1$ cannot be replaced by $\text{dist}(p_n, U) = o(1)$ and $|p_{n+1} - p_n| = o(1)$. It is not clear whether the conditions that $\text{dist}(p_n, U) = o(|p_n|)$ and $|p_{n+1}/p_n| \to 1$ can be improved.

If information about the asymptotics of f in a Baker domain U is given, then one can sometimes use this to prove that U contain at least one singular value. Hinkkanen [38, Theorem 2] obtained such a result for functions f satisfying $f(z) = z + az^{-m} + O(z^{-m-\delta})$ as $z \to \infty$ in a suitable sector, where $a \in \mathbb{C}\backslash\{0\}$ and $\delta > 0$. In some cases the asymptotic behavior in a Baker domain domain U even implies the existence of infinitely many singular values in U. The following result is due to Rippon and Stallard [49, Theorem 2]

Theorem 8 *Suppose that*

$$f(z) = az + bz^k e^{-z}(1 + o(1))$$

as $\text{Re}\, z \to \infty$, where $a > 1, b > 0, k \in \mathbb{N}$. Then there exists $\rho > 0, R > 0$ such that $\{z : |z^k e^{-z}| < \rho, |z| > R\}$ is contained in a Baker domain U, and U contains infinitely many singular values.

Rippon and Stallard [49] have also given some other conditions implying that a Baker domain contains infinitely many singular values.

The following theorem [13] is a further result in this direction. In order to state it, we denote by λ_U the hyperbolic metric in a domain U and consider for a Baker domain U and $z \in U$ the sequence $(\rho_n(z))$ defined by $\rho_n(z) := \lambda_U(f^{n+1}(z), f^n(z))$. By Schwarz's lemma, $(\rho_n(z))$ is non-increasing so that $\rho(z) := \lim_{n\to\infty} \rho_n(z)$ exists. The sequences $(\rho_n(z))$ were also considered by Rippon and Stallard [49] in their proof of Theorem 8, and they have also been studied by Bargmann [8], Bonfert [24] and König [41].

Theorem 9 *If U is an invariant Baker domain such that $U \cap \text{sing}(f^{-1})$ is bounded, then we have one of the following three cases:*

(i) $\inf_{z\in U} \rho(z) > 0$,

(ii) $\rho(z) > 0$ *for all* $z \in U$, *but* $\inf_{z\in U} \rho(z) = 0$, *and there exists* $a \geq 0$ *such that*

$$\rho_n(z) = \rho(z) + a \cdot \frac{1}{n^3} + O\left(\frac{1}{n^4}\right),$$

(iii) $\rho(z) = 0$ *for all* $z \in U$, *and there exists* $b \in \mathbb{R}$ *such that*

$$\rho_n(z) = \frac{1}{2n} + b \cdot \frac{\log n}{n^2} + O\left(\frac{1}{n^2}\right).$$

The main idea in the proof is to use that U is simply connected [2, Theorem 1] and that if $\varphi : \mathbb{D} \to U$ is simply-connected, then $g := \phi^{-1} \circ f \circ \phi$ is an inner function. This approach has been used in a number of papers [4, 5, 8, 40, 41]. It turns out that under the hypotheses of Theorem 9 the function g can be continued across some arc on $\partial\mathbb{D}$ and it has a fixed point ξ on this arc. If ξ is attracting, then we have case (i). Otherwise ξ is a multiple fixed point of multiplicity 2 or 3, and this leads to cases (iii) and (ii). The numbers a and b occurring in (ii) and (iii) can be expressed in terms of g and ξ. More precisely, we have

$$a = \frac{\iota(g,\xi) - 1}{3\tanh\left(\frac{\rho(z)}{2}\right)} \quad \text{and} \quad b = \frac{\iota(g,\xi)}{4} - \frac{3}{8}.$$

As an example where Theorem 9 applies we mention entire functions f which satisfy $f(z) = z + c + o(1)$ as $|z| \to \infty$ in some sector $\{z : |\arg z| \leq \eta\}$, where $c, \eta > 0$. It is easy to see that such a function f has an invariant Baker domain U containing $\{z : |\arg z| \leq \eta, \text{Re } z > R\}$ for some $R > 0$. Using Theorem 9 one can show that if f has finite order, then $U \cap \text{sing}(f^{-1})$ is unbounded; see [13, Theorem 3].

Finally we mention that for functions f satisfying $f(z) = z + c + o(1)$ as $|z| \to \infty$ in some sector $\{z : |\arg z| \leq \eta\}$, where $c, \eta > 0$, there is also an analogue of Theorem 2 for Baker domains; see also [15] besides [14, 25] for this result.

Theorem 10 *Suppose that* $f(z) = z + 1 + c/z + o(1/z)$ *as* $|z| \to \infty$ *in some sector* $\{z : |\arg z| \leq \eta\}$, *where* $c \in \mathbb{C}$ *and* $\eta > 0$. *Then* f *has a Baker domain* U *containing* $\{z : |\arg z| \leq \eta, \mathrm{Re}\, z > R\}$ *for some* $R > 0$. *If* $\mathrm{Re}\, c < \frac{3}{20}$, *then* U *contains at least two singular values.*

In order to see the analogy to Theorem 2 we note that if f is a rational function satisfying $f(z) = z + 1 + c/z + o(1/z)$ as $|z| \to \infty$, then f has a fixed point of multiplicity 2 at ∞, with $\iota(f, \infty) = 1 - c$. Thus the condition $\mathrm{Re}\, c < \frac{3}{20}$ corresponds to (2) if $m = 1$.

Note that the hypothesis on c is satisfied in particular if $c = 0$.

2.4. CONCLUDING REMARKS

Here we have concentrated on relations between periodic Fatou components and singular values. We mention that there are also relations between wandering domains and singular values. This concerns results about the non-existence of wandering domains [11, 19, 30, 34, 55], which extend the result of Sullivan [57] that rational functions do not have wandering domains to certain classes of transcendental entire functions, as well as results on the limit functions of the iterates in wandering domains [1, 16, 17].

There are many other results in complex dynamics where the singular values play an important role. For example, there are intimate relations between singular values and the geometry of the Julia set, its Hausdorff dimension, and various other properties. We refer the reader to the textbooks and articles mentioned in the introduction.

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