ZEROS OF SOLUTIONS OF A FUNCTIONAL EQUATION

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Dedicated to the memory of Professor Dieter Gaier, collaborator and true friend.

ABSTRACT. We consider the zeros of transcendental entire solutions f of the functional equation $\sum_{j=0}^{m} a_j(z) f(c^j z) = Q(z)$, where $c \in \mathbb{C}$, $0 < |c| < 1$, and Q and the a_j are polynomials. Under a suitable hypothesis concerning the associated Newton-Puiseux diagram we show that the zeros of f are asymptotic to certain geometric progressions. More precisely, with this hypothesis there exist positive integers M and N such that the zero set can written in the form $\{z_{n,\mu} : \mu \in \{1, 2, \ldots, M\}, n \in \mathbb{N}\}$ where for each μ in $\{1, 2, \ldots, M\}$ there exists A_μ in $\mathbb{C} \setminus \{0\}$ with $z_{n,\mu} \sim A_\mu \, c^{-Nn}$ as $n \to \infty$. The proof is achieved by showing that f behaves asymptotically like a product of θ-functions. The hypothesis on the Newton-Puiseux diagram is satisfied, e. g., if for each positive σ and each real τ the line $\{(x, y) \in \mathbb{R}^2 : y = \sigma x + \tau\}$ contains at most two points of the form $(j, \deg(a_j))$. In particular, this is the case if all a_j are linear, in which case the above conclusion follows with $M = 1$ which means that the zeros are asymptotic to only one geometric progression. The hypothesis on the Newton-Puiseux diagram is also satisfied if $m = 1$. If $m = 1$ and $Q \equiv 0$, however, we have a much simpler and more precise result. We illustrate our results by a number of examples. In particular, we show that if the hypothesis on the Newton-Puiseux diagram is not satisfied, then the zeros of the solutions need not be asymptotic to a finite number of geometric progessions.

1. Introduction and main results

The functional equation

$$\sum_{j=0}^{m} a_j(z) f(c^j z) = Q(z), \tag{1.1}$$

where $c \in \mathbb{C}$, $0 < |c| < 1$, and Q and the a_j, $j = 0, \ldots, m$ are polynomials, has been studied by various authors; see, e. g., [2, 3, 4, 7, 10, 11].

In a letter to the second author Mourad Ismail has conjectured that, at least in some cases, the zeros z_n of a transcendental entire solution of (1.1), arranged in order of increasing moduli, satisfy

$$z_n = c_1 q^{-n/a} + c_2 q^{bn} \left(1 + \mathcal{O}\left(q^{cn}\right)\right)$$

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for some constants c_1, c_2, a, b, c such that a, b, c are positive. This conjecture was motivated by his results on the q-Bessel functions $J_\nu^{(2)}(z;q)$ obtained in [8].

In this paper we consider some special cases, which show that such a result at least in weakened form is true in some cases, but not in others.

We begin with the rather simple case that $m = 1$ and $Q \equiv 0$.

Theorem 1. *Suppose that f is a transcendental entire solution of the equation*

$$f(z) - a(z)f(cz) = 0, \tag{1.2}$$

where $0 < |c| < 1$ and a is a polynomial of degree d. Then there exists a non-negative integer p and $A, z_1, \ldots, z_d$ in $\mathbb{C} \setminus \{0\}$ such that

$$f(z) = Az^p \prod_{\mu=1}^{d} \prod_{j=0}^{\infty} \left(1 - \frac{c^j z}{z_\mu}\right). \tag{1.3}$$

Evidently the function f given by (1.3) satisfies (1.2) for

$$a(z) = c^{-p} \prod_{\mu=1}^{d} \left(1 - \frac{z}{z_\mu}\right). \tag{1.4}$$

We note that, if $d = 0$ so that a is constant, then the only solutions of (1.2) are monomials $f(z) = Az^p$, so that there are no transcendental solutions.

To state our results in a second case we recall some results about the growth of entire solutions of (1.1) obtained in [3, 10]. Write $p_j := \deg(a_j)$ and $d_j := p_j - p_0$. The *Newton-Puiseux diagram*, P, is defined as the convex hull of

$$\bigcup_{j=0}^{m} \left\{(x, y) \in \mathbb{R}^2 : x \geq j \text{ and } y \leq d_j\right\}.$$

Let (j_k, d_{j_k}), $k = 0, \ldots, K$, be the vertices of P, where

$$0 = j_0 < j_1 < \cdots < j_K \leq m.$$

We note that if (1.1) has a transcendental entire solution, then there exists j in $\{1, \ldots, m\}$ such that $d_j > 0$; see [3, Theorem 1.1]. This implies that $K \geq 1$. For $k = 1, \ldots, K$ we define

$$\sigma_k := \frac{d_{j_k} - d_{j_{k-1}}}{j_k - j_{k-1}}.$$

Then $\sigma_1 > \sigma_2 > \cdots > \sigma_K > 0$. The σ_k are the slopes of the segments which form the boundary of P.

It was shown in [3, Theorem 1.2] and [10] that if f is an entire transcendental solution of (1.1), then there exists k in $\{1, \ldots K\}$ such that

$$\log M(r, f) \sim \frac{\sigma_k}{-2\log|c|} (\log r)^2 \tag{1.5}$$

as $r \to \infty$. This generalized earlier results of Valiron [11, §54] and Wittich [12]. We note that Ishizaki and Yanagihara [7] have shown that if $a_0 \equiv 1$ and $Q \equiv 0$, and if the equation (1.1) is "irreducible," then $k = 1$ in (1.5). In the general case, however, it is also possible that $k > 1$.

Here we shall obtain asymptotic expansions for the coefficients and zeros of f under the additional assumption that the segment of the boundary of P whose slope is σ_k contains no point (j, d_j) except for its endpoints $(j_{k-1}, d_{j_{k-1}})$ and (j_k, d_{j_k}). We note that this additional hypothesis is clearly satisfied if $j_k - j_{k-1} = 1$ or $d_{j_k} - d_{j_{k-1}} = 1$.

We define $M := d_{j_k} - d_{j_{k-1}}$ and $N := j_k - j_{k-1}$ so that $\sigma_k = \frac{M}{N}$. We also define $\rho := c^{1/(2M)}$, for some fixed branch of the root.

Theorem 2. *Let $c, Q, a_j, d_j, j_k, \sigma_k, M, N$ and ρ be as above, and let*

$$f(z) = \sum_{n=0}^{\infty} \alpha_n z^n \tag{1.6}$$

be a transcendental entire solution of (1.1) *satisfying* (1.5). *Suppose that the segment from $(j_{k-1}, d_{j_{k-1}})$ to (j_k, d_{j_k}) does not contain any point (j, d_j) where $j_{k-1} < j < j_k$.*

Then there exist $(\eta_0, \eta_1, \ldots, \eta_{M-1})$ in $\mathbb{C}^M \setminus \{(0,0,\ldots,0)\}$ and b in $\mathbb{C} \setminus \{0\}$ such that, for each r in $\{0, 1, \ldots, M-1\}$,

$$\alpha_n = \rho^{Nn^2} b^n \left(\eta_r + \mathcal{O}\left(\rho^{2n}\right)\right) \tag{1.7}$$

as $n \to \infty$ while satisfying $n \equiv r (\mathrm{mod}\ M)$.

Moreover, the set of zeros of f can be written in the form

$$\{z_{n,\mu} : \mu \in \{1, 2, \ldots, M\}, n \in \mathbb{N}\}$$

such that for each μ there exists A_μ in $\mathbb{C} \setminus \{0\}$ with $z_{n,\mu} \sim A_\mu\, c^{-Nn}$ as $n \to \infty$. More precisely, if M_μ denotes the cardinality of the set of all λ in $\{1, 2, \ldots, M\}$ for which there exists an integer ν with $A_\lambda = c^{N\nu} A_\mu$, then

$$z_{n,\mu} = A_\mu\, c^{-Nn} \left(1 + \mathcal{O}\left(c^{n/M_\mu}\right)\right) \tag{1.8}$$

as $n \to \infty$. In particular,

$$z_{n,\mu} = A_\mu\, c^{-Nn} \left(1 + \mathcal{O}\left(\rho^{2n}\right)\right). \tag{1.9}$$

We shall see in Example 2 in §6 that the term $\mathcal{O}(\rho^{2n})$ in (1.7) and (1.9) cannot be replaced by $o(\rho^{2n})$.

Already now we give an example to show that the hypothesis made on the Newton-Puiseux diagram is necessary in Theorem 2.

Example 1. Suppose that $\eta \in (0,1) \setminus \{\frac{1}{2}\}$ and define $\gamma := \cos \eta\pi$ and $t_{1,2} := e^{\pm i\eta\pi}$. Then $t_{1,2}$ are the solutions of $t^2 - 2\gamma t + 1 = 0$. Suppose that $c_1, c_2 \in \mathbb{C}$ and let f be defined by (1.6) with

$$\alpha_n := c_1 t_1^n \rho^{2n^2} + c_2 t_2^n \rho^{2n^2}.$$

With $c := \rho^4$ and

$$Q(z) := \left(-(c_1+c_2)\gamma\rho^2 + i(c_1-c_2)\rho^2\sqrt{1-\gamma^2}\right) z + c_1 + c_2$$

we find that f satisfies the equation

$$f(z) - 2\gamma\rho^2 z f(cz) + c^2 z^2 f(c^2 z) = Q(z).$$

Thus f satisfies an equation of the form (1.1) with $m = 2$ and $\deg(a_j) = j$ for j in $\{0, 1, 2\}$. The vertices of the Newton-Puiseux diagram are $(0, 0)$ and $(2, 2)$ so that $M = N = 2$. Thus f satisfies (1.5) with $k = 1$ and $\sigma_1 = 1$. This also follows directly from Lemma 1 in §3.2.

We note that the point $(1, \deg(a_1))$ lies on the segment connecting the vertices $(0, 0)$ and $(2, 2)$. Thus the hypothesis of Theorem 2 is not satisfied. In fact, it is not difficult to see that unless η is rational so that $t_{1,2}$ are roots of unity, or if $c_1 = 0$ or $c_2 = 0$, then α_n does not have asymptotics given by (1.7). And even if η is rational, then the asymptotics given by (1.7) are, in general, satisfied only for some M greater than 2.

Example 1 shows that the hypothesis made on the Newton-Puiseux diagram is necessary for (1.7). Next we show that if this hypothesis is not satisfied, then the zeros of the solutions need not be asymptotic to any finite number of geometric progessions. This follows from the following result.

Theorem 3. *Let $\eta, \gamma, t_{1,2}, c_{1,2}, \rho, c$ and f be as in Example 1. Suppose that $\rho > 0$ and that η is irrational. If $c_1, c_2 \neq 0$ and $|c_1| \neq |c_2|$, then the arguments of the zeros of f are dense in some interval contained in $[-\pi, \pi]$, but they are not dense in $[-\pi, \pi]$.*

An important special case of Theorem 2 is when all the a_j in (1.1) are linear. Then $k = K = 1$, $j_1 = \min\{j : a_j \text{ is not constant}\}$, $d_{j_0} = 0$, $d_{j_1} = 1$, and thus f satisfies (1.5) with $k = 1$ and $\sigma_1 = 1/j_1$. We deduce that (1.7) and (1.9) hold with $\rho = \sqrt{c}$ and $N = j_1$. We note that our proof simplifies in the case where all the a_j are linear, cf. the remark at the end of §3.2.

2. Proof of Theorem 1

We suppose that $f(z) = Az^p + \mathcal{O}(z^{p+1})$ is a transcendental entire solution of (1.2). We write $f_p(z) = f(z)/Az^p$ so that

$$f_p(z) = 1 + \sum_{n=1}^{\infty} \alpha_n z^n. \tag{2.1}$$

Now (1.2) yields

$$Az^p f_p(z) = a(z) A c^p z^p f_p(cz)$$

so that

$$f_p(z) = a_p(z) f_p(cz)$$

with $a_p(z) = c^p a(z)$. It follows that $a_p(0) = 1$ and hence $a(0) = c^{-p}$. We deduce that if $z_1, \ldots, z_d$ are the zeros of a, then a takes the form (1.4).

It follows from (1.2) that the zeros $z_1, \dots, z_d$ of a are also zeros of f, and induction shows that $c^{-j} z_\mu$ is also a zero of f if $j \in \mathbb{N}$ and $\mu \in \{1, \dots, d\}$, with the multiplicity counted correspondingly if z_μ is a multiple zero of a.

We denote the righthand side of (1.3) by $g(z)$. Then (1.2) holds with f replaced by g, and thus $h = f/g$ satisfies

$$h(z) = h(cz). \tag{2.2}$$

We have $h(0) = 1$, say $h(z) = 1 + \sum_{j=1}^{\infty} \beta_n z^n$ and equating coefficients in (2.2) we find $\beta_n = c^n \beta_n$. This implies that $\beta_n = 0$ for $n \geq 1$. Thus $h(z) \equiv 1$ so that $f = g$. □

3. Proof of Theorem 2. Part I: Asymptotics of the coefficients

3.1. **A recurrence relation for the coefficients.** Let f be a transcendental entire solution of (1.1) with Taylor series given by (1.6). Write

$$a_j(z) = \sum_{i=0}^{p_j} b_{j,i} z^i$$

with $b_{j,p_j} \neq 0$. We define $b_{j,i} = 0$ for $i > p_j$, and we write $p := \max_{j \in \{0,\dots,n\}} p_j$. Comparing coefficients in (1.1) we find for $n > \max\{p, \deg(Q)\}$ that

$$\sum_{i=0}^{p} \alpha_{n-i} \sum_{j=0}^{m} b_{j,i} c^{j(n-i)} = 0.$$

With

$$\beta_n := \rho^{-Nn^2} \alpha_n$$

and

$$c_{j,i} := b_{j,i} \rho^{Ni^2} c^{-ji} = b_{j,i} \rho^{Ni^2 - 2Mji} \tag{3.1}$$

we deduce that

$$0 = \sum_{i=0}^{p} \rho^{N(n-i)^2} \beta_{n-i} \sum_{j=0}^{m} b_{j,i} c^{j(n-i)} = \rho^{Nn^2} \sum_{i=0}^{p} \beta_{n-i} \sum_{j=0}^{m} c_{j,i} \rho^{(Mj-Ni)2n}.$$

Putting

$$u_{n,i} := \sum_{j=0}^{m} c_{j,i} \rho^{(Mj-Ni)2n}$$

we thus have

$$\sum_{i=0}^{p} \beta_{n-i} u_{n,i} = 0. \tag{3.2}$$

From the definition of σ_k we have

$$d_j \leq d_{j_k} + \sigma_k (j - j_k) = d_{j_{k-1}} + \sigma_k (j - j_{k-1}), \tag{3.3}$$

if $j \in \{0, 1, \ldots, m\}$, with strict inequality if $j < j_{k-1}$ or $j > j_k$. By hypothesis we also have strict inequality if $j_{k-1} < j < j_k$. We may write (3.3) in the form

$$d_j - \sigma_k j \leq d_{j_k} - \sigma_k j_k = d_{j_{k-1}} - \sigma_k j_{k-1}$$

and thus, recalling that $p_j = \deg(a_j) = d_j + p_0$ and $\sigma_k = M/N$, we obtain

$$Np_j - Mj \leq Np_{j_k} - Mj_k = Np_{j_{k-1}} - Mj_{k-1} =: L,$$

with strict inequality if $j \neq j_{k-1}$ and $j \neq j_k$. It follows that

$$L + Mj - Np_j \geq 0,$$

with strict inequality if $j \neq j_{k-1}$ and $j \neq j_k$. For $0 \leq j \leq m$ and $0 \leq i \leq p_j$ we thus have

$$L + Mj - Ni \geq 0,$$

with strict inequality if $(j, i) \neq (j_{k-1}, p_{j_{k-1}})$ and $(j, i) \neq (j_k, p_{j_k})$. We now put

$$v_{n,i} := \rho^{2Ln} u_{n,i} = \sum_{j=0}^{m} c_{j,i} \rho^{(L+Mj-Ni)2n}. \tag{3.4}$$

Since $c_{j,i} = b_{j,i} = 0$ for $i > p_j$ the above considerations show that all powers of ρ occurring on the right hand side of (3.4) are positive, except if $(j, i) = (j_{k-1}, p_{j_{k-1}})$ or $(j, i) = (j_k, p_{j_k})$. With $\ell := p_{j_{k-1}}$ so that $p_{j_k} = p_{j_{k-1}} + M = \ell + M$ it follows that

$$v_{n,i} = \mathcal{O}\left(\rho^{2n}\right) = \mathcal{O}\left(|c|^{n/M}\right)$$

for $i \neq \ell, \ell + M$ while

$$v_{n,\ell} = c_{j_{k-1},\ell} + \mathcal{O}\left(\rho^{2n}\right)$$

and

$$v_{n,\ell+M} = c_{j_k,\ell+M} + \mathcal{O}\left(\rho^{2n}\right).$$

By (3.2) and (3.4) we have $\sum_{i=0}^{p} \beta_{n-i} v_{n,i} = 0$. We define

$$v^*_{n,i} := v_{n,i}$$

for $i \neq \ell, \ell + M$,

$$v^*_{n,\ell} := v_{n,\ell} - c_{j_{k-1},\ell}$$

and

$$v^*_{n,\ell+M} := v_{n,\ell+M} - c_{j_k,\ell+M}.$$

Then $v^*_{n,i} = \mathcal{O}(\rho^{2n})$ as $n \to \infty$, for all i in $\{1, \ldots, p\}$, and

$$c_{j_{k-1},\ell}\beta_{n-\ell} + c_{j_k,\ell+M}\beta_{n-\ell-M} + \sum_{i=0}^{p} \beta_{n-i} v^*_{n,i} = 0.$$

Now $b_{j_{k-1},\ell}$ and $b_{j_k,\ell+M}$ are the leading coefficients of the polynomials $a_{j_{k-1}}$ and a_{j_k} respectively and so are not zero, so by (3.1) $c_{j_{k-1},\ell}$ and $c_{j_k,\ell+M}$ are not zero either.

We now choose b such that

$$b^M = -\frac{c_{j_k,\ell+M}}{c_{j_{k-1},\ell}}.$$

Using (3.1) and $N = j_k - j_{k-1}$ we deduce that

$$\begin{aligned} -\frac{b_{j_{k-1},\ell}}{b_{j_k,\ell+M}} b^M &= \rho^{N((\ell+M)^2-\ell^2)-2M\ell(j_k-j_{k-1})-2M^2 j_k} \\ &= \rho^{M^2(N-2j_k)} \\ &= \rho^{-M^2(j_{k-1}+j_k)}. \end{aligned} \tag{3.5}$$

We also define

$$\gamma_n := b^{-n}\beta_n = \rho^{-Nn^2} b^{-n}\alpha_n. \tag{3.6}$$

Then

$$\gamma_n = \gamma_{n-M} + \sum_{i=\ell-p}^{\ell} w_{n,i}\gamma_{n+i} \tag{3.7}$$

with $w_{n,i} = \mathcal{O}(\rho^{2n})$ as $n \to \infty$. More precisely, $w_{n,i}$ is a polynomial in ρ^{2n} with coefficients depending only on i and no constant term.

3.2. **The asymptotic behaviour of the coefficients.** We note that in order to obtain (3.7) we have not used (1.5) yet. To make use of (1.5) we shall need the following lemma, which is a special case of a result of Juneja, Kapoor and Bajpai [9, Theorem 1].

Lemma 1. *Let f be a transcendental entire function with Taylor series given by* (1.6). *Then*

$$\limsup_{r\to\infty} \frac{\log M(r,f)}{(\log r)^2} = \frac{1}{4} \limsup_{n\to\infty} \frac{n^2}{-\log|\alpha_n|}.$$

It follows from Lemma 1 and (1.5) that

$$\frac{1}{4} \limsup_{n\to\infty} \frac{n^2}{-\log|\alpha_n|} = \frac{\sigma_k}{-2\log|c|} = -\frac{M}{2N\log|c|}.$$

Thus

$$\limsup_{n\to\infty} \frac{\log|\alpha_n|}{n^2} = \frac{N\log|c|}{2M}.$$

(Here we put $\log|\alpha_n|/n^2 = -\infty$ if $\alpha_n = 0$. This corresponds to $n^2/\log|\alpha_n| = 0$ in Lemma 1 if $\alpha_n = 0$.) Since

$$\begin{aligned} \log|\gamma_n| &= \log|\beta_n| + \mathcal{O}(n) \\ &= \log|\alpha_n| - Nn^2\log|\rho| + \mathcal{O}(n) \\ &= \log|\alpha_n| - \frac{N\log|c|}{2M} n^2 + \mathcal{O}(n) \end{aligned}$$

we obtain

$$\limsup_{n\to\infty} \frac{\log|\gamma_n|}{n^2} = 0. \tag{3.8}$$

Lemma 2. *Suppose that $\ell, p, M \in \mathbb{Z}$, $0 \le \ell < p$, $1 \le M \le p-\ell$, and $0 < s < 1$. For $\ell - p \le i \le \ell$, let $(w_{n,i})_{n\in\mathbb{N}}$ be a sequence of complex numbers which satisfies $|w_{n,i}| = \mathcal{O}(s^n)$ as $n \to \infty$. Let (γ_n) be a sequence of complex numbers satisfying (3.7) and (3.8). Then*

$$\eta_r := \lim_{n\to\infty} \gamma_{nM+r}$$

exists for all r in $\{0,1,\ldots,M-1\}$. Moreover,

$$|\gamma_{nM+r} - \eta_r| = \mathcal{O}(s^{Mn})$$

as $n \to \infty$ for all r, and there exists r in $\{0,1,\ldots,M-1\}$ with $\eta_r \ne 0$.

Proof. For abbreviation we put $c_n := |\gamma_n|$. We choose a positive constant C such that $|w_{n,i}| < Cs^n$ for all n and i.

First we claim that for each K greater than 1 there exists n_0 in $\mathbb{N}$ such that

$$c_n \le K^n \max\{c_{n-1}, c_{n-2}, \ldots, c_{n-p}\} \tag{3.9}$$

for $n \ge n_0$.

To prove this claim we may assume that $1 < K < s^{-1/(\ell+1)}$. We consider

$$I := \{n > p : c_n > K^n \max\{c_{n-1}, c_{n-2}, \ldots, c_{n-p}\}\}.$$

We have to show that I is bounded.

First we assume that there exists an increasing sequence (n_k) in I such that $n_{k+1} \le n_k + p$ for all k. Then $n_k \le n_1 + (k-1)p = kp + n_1 - p$ and

$$\begin{aligned} \log c_{n_k} &> n_k \log K + \log\max\{c_{n_k-1}, \ldots, c_{n_k-p}\} \\ &\ge n_k \log K + \log c_{n_{k-1}}. \end{aligned}$$

Induction yields

$$\log c_{n_k} > \left(\sum_{j=2}^{k} n_j\right) \log K + \log c_{n_1}.$$

Since $n_j \ge j$ we have for large k

$$\sum_{j=2}^{k} n_j \ge \sum_{j=2}^{k} j > \frac{1}{2}k^2 \ge \frac{1}{2}\left(\frac{n_k - n_1 + p}{p}\right)^2 > \frac{1}{3}\left(\frac{n_k}{p}\right)^2.$$

Hence we find that

$$\log c_{n_k} > \left(\sum_{j=2}^{k} n_j\right) \log K + \log c_{n_1} > \frac{\log K}{4p^2} n_k^2$$

for large k, contradicting (3.8). Thus a sequence (n_k) as above does not exist. This implies that, if I is unbounded, then there exist arbitrarily large n in I such that

$$\{n+1, n+2, \ldots, n+p\} \cap I = \emptyset.$$

For such n in I we have

$$c_{n+1} \leq K^{n+1} \max\{c_n, \ldots, c_{n-p+1}\} = K^{n+1} c_n$$

and hence (if $p \geq 2$ so that $n+2 \notin I$) also

$$\begin{aligned} c_{n+2} &\leq K^{n+2} \max\{c_{n+1}, \ldots, c_{n-p+2}\} \\ &\leq K^{n+2} K^{n+1} \max\{c_n, \ldots, c_{n-p+1}\} \\ &= K^{(n+2)+(n+1)} c_n. \end{aligned}$$

Since $\ell \leq p$ we obtain inductively $c_{n+k} \leq K^{kn+\frac{1}{2}k(k+1)} c_n$ for $1 \leq k \leq \ell$. If n in I as above is sufficiently large we thus obtain

$$c_{n+k} \leq K^{(\ell+1)n} c_n \tag{3.10}$$

for $1 \leq k \leq \ell$. It follows from the the definition of I that (3.10) also holds for $\ell - p \leq k \leq 0$. (In fact a much stronger inequality than (3.10) holds for these values of k.) Since $n \in I$ we have $c_{n-M} < K^{-n} c_n$. Substituting this and (3.10) in (3.7) yields

$$\begin{aligned} c_n &\leq c_{n-M} + \sum_{k=\ell-p}^{\ell} |w_{n,k}| c_{n+k} \\ &\leq K^{-n} c_n + \sum_{k=\ell-p}^{\ell} C s^n K^{(\ell+1)n} c_n \\ &= c_n \left(K^{-n} + (p+1) C \left(s K^{\ell+1} \right)^n \right). \end{aligned}$$

This is a contradiction for large n, since $sK^{\ell+1} < 1$. Thus (3.9) holds for sufficiently large n, provided that $K > 1$.

We now proceed to use (3.9) to obtain an inequality stronger than (3.9). To do this we note that (3.9) and an inductive argument like the one used to obtain (3.10) yield

$$c_{n+k} \leq K^{(\ell+1)n} \max\{c_{n-1}, c_{n-2}, \ldots, c_{n-p}\}$$

for $K > 1$ and $0 \leq k \leq \ell$, provided that n is sufficiently large. Together with (3.7) this implies that

$$\begin{aligned} c_n &\leq c_{n-M} + \sum_{k=\ell-p}^{\ell} |w_{n,k}| c_{n+k} \\ &\leq \left(1 + (p+1) C s^n K^{(\ell+1)n}\right) \max\{c_{n-1}, c_{n-2}, \ldots, c_{n-p}\}. \end{aligned}$$

With $\tau := (p+1)C$ and $t := sK^{\ell+1}$ we thus have

$$c_n \leq (1 + \tau t^n) \max\{c_{n-1}, c_{n-2}, \ldots, c_{n-p}\} \tag{3.11}$$

for large n. By our choice of K we have $t < 1$. Thus (3.11) is an improvement of (3.9) for large n.

Since $\prod_{n=1}^{\infty}(1+\tau t^n) < \infty$, it follows from (3.11) that the sequence (c_n) is bounded, say $c_n \le A$, if $n \in \mathbb{N}$. Combined with (3.7) this implies that

$$|\gamma_n - \gamma_{n-M}| \le \sum_{k=\ell-p}^{\ell} |w_{n,k}| c_{n+k} \le (p+1) C s^n A$$

for $n > M$. We conclude that

$$\eta_r := \lim_{n\to\infty} \gamma_{nM+r} = \gamma_r + \sum_{j=1}^{\infty} \left(\gamma_{jM+r} - \gamma_{(j-1)M+r}\right)$$

exists for each r in $\{0, 1, \ldots, M-1\}$, and

$$\eta_r - \gamma_{nM+r} = \sum_{j=n+1}^{\infty} \left(\gamma_{jM+r} - \gamma_{(j-1)M+r}\right) = \mathcal{O}(s^{Mn})$$

as $n \to \infty$.

It remains to show that $\eta_r \neq 0$ for some r. To do this we assume that $\eta_r = 0$ for all r, and seek a contradiction. We first note that then $A_n := \max_{k\ge n} c_k \to 0$ as $n \to \infty$. We obtain

$$\begin{aligned} c_n &= \left| \sum_{j=1}^{\infty} \left(\gamma_{n+jM} - \gamma_{n+(j-1)M}\right) \right| \\ &\le \sum_{j=1}^{\infty} \sum_{k=\ell-p}^{\ell} |w_{n+jM,k}| c_{n+jM+k} \\ &\le \sum_{j=1}^{\infty} \sum_{k=\ell-p}^{\ell} C s^{n+jM} A_{n+jM+k} \\ &\le (p+1) C s^n A_{n-p} \sum_{j=1}^{\infty} s^{jM} \\ &= \frac{(p+1)Cs^M}{1-s^M} s^n A_{n-p}. \end{aligned}$$

With $B := (p+1)Cs^M/(1-s^M)$ we thus have

$$A_n \le B s^n A_{n-p}$$

for $n \ge p$. Induction yields

$$A_n \le B^k s^{kn-\frac{1}{2}(k-1)kp} A_{n-kp}$$

if $k \in \mathbb{N}$ and $kp < n$. Choosing k with $n - p \le kp < n$ we obtain

$$kn - \frac{1}{2}(k-1)kp = k\left(n - \frac{1}{2}(k-1)p\right) \ge \left(\frac{n}{p} - 1\right)\left(n - \frac{1}{2}(n-p)\right) = \frac{n^2}{2p} - \frac{p}{2}$$

and thus

$$\log|\gamma_n| \le \log A_n \le \frac{n}{p}\max\{0,\log B\} + \left(\frac{n^2}{2p} - \frac{p}{2}\right)\log s + \log A_1.$$

This yields

$$\limsup_{n\to\infty} \frac{\log|\gamma_n|}{n^2} \le \frac{\log s}{2p} < 0,$$

contradicting (3.8). This completes the proof of Lemma 2. □

Remark. If all the a_j are linear, then $M = 1$, and we have $p = 1$ and $\ell = 0$ in §3.1. This simplifies the proof of Lemma 2 considerably. The arguments in §4 are also much simpler if $M = 1$.

3.3. **Completion of the proof of (1.7).** Let f, α_n be as in the statement of Theorem 2 and β_n, γ_n as in §3.1. From (3.7), (3.8) and Lemma 2 we deduce with $s = \rho^2$ that $\gamma_n = \eta_r + \mathcal{O}(s^n)$ as $n \to \infty$ satisfying $n \equiv r (\mathrm{mod}\ M)$. Using (3.6) we deduce that

$$\alpha_n = \rho^{Nn^2}\beta_n = \rho^{Nn^2} b^n \gamma_n = \rho^{Nn^2} b^n \left(\eta_r + \mathcal{O}(s^n)\right),$$

as $n \to \infty$ satisfying $n \equiv r(\mathrm{mod}\ M)$. Since $s = \rho^2$, (1.7) follows. □

4. Proof of Theorem 2. Part II: Asymptotics of the zeros

4.1. **A product of theta functions.** We shall prove (1.8) by comparing f with

$$F(z) := \sum_{n=-\infty}^{\infty} \eta_{r_n} \rho^{Nn^2} b^n z^n, \tag{4.1}$$

where $\eta_0, \dots, \eta_{M-1}$ are chosen according to the first part of the theorem, and $r_n \in \{0, 1, \dots, M-1\}$ with $n \equiv r_n(\mathrm{mod}\ M)$. A computation shows that

$$F(z) = z^M b^M \rho^{NM^2} F(c^N z).$$

With

$$A := b^M \rho^{NM^2} \tag{4.2}$$

the equation for F takes the form

$$F(z) = A z^M F(c^N z). \tag{4.3}$$

We shall express F as a product of θ-functions. The θ-function

$$\theta(z,q) := \sum_{n=-\infty}^{\infty} q^{n^2} z^n \tag{4.4}$$

is defined for $|q| < 1$ and $z \in \mathbb{C}\setminus\{0\}$. Jacobi's triple product identity (see, e. g., [1, Theorem 2.8]) says that

$$\theta(z,q) = \omega \prod_{n=1}^{\infty} \left\{ \left(1 + \frac{q^{2n-1}}{z}\right)\left(1 + q^{2n-1} z\right) \right\} \tag{4.5}$$

where

$$\omega = \prod_{n=1}^{\infty} \left(1 - q^{2n}\right).$$

Theorem 4. *Suppose that F is holomorphic in $\mathbb{C} \setminus \{0\}$ and satisfies the functional equation* (4.3)*. Suppose also that $q \in \mathbb{C}$ with $q^2 = c^N$. Then there exist $z_1, z_2, \ldots, z_M$ and C in $\mathbb{C} \setminus \{0\}$ such that*

$$F(z) = C \prod_{\mu=1}^{M} \theta\left(-\frac{z}{q z_\mu}, q\right). \tag{4.6}$$

Further

$$\prod_{\mu=1}^{M} z_\mu = \frac{(-1)^M}{A}. \tag{4.7}$$

Proof. It follows from (4.3) that if z_μ is a zero of F, then so is $c^{N\nu} z_\mu$ for each integer ν. Thus the zeros of F come in geometric progressions. Suppose that r_1, r_2 are positive numbers such that $r_1 = |c|^N r_2$ and F has no zeros on $|z| = r_1$ and $|z| = r_2$. We shall show that F has M zeros in the annulus $r_1 < |z| < r_2$, counting multiplicities. To do so we note that by the argument principle the number of zeros of F in the annulus $r_1 < |z| < r_2$ is given by

$$\begin{aligned} &\frac{1}{2\pi i}\left(\int_{|z|=r_2} \frac{F'(z)}{F(z)} dz - \int_{|z|=r_1} \frac{F'(z)}{F(z)} dz\right) \\ = \; &\frac{1}{2\pi}\left(\int_0^{2\pi} \frac{r_2 e^{i\theta} F'(r_2 e^{i\theta})}{F(r_2 e^{i\theta})} d\theta - \int_0^{2\pi} \frac{r_1 e^{i\theta} F'(r_1 e^{i\theta})}{F(r_1 e^{i\theta})} d\theta\right). \end{aligned}$$

Now (4.3) yields

$$\log F(z) = \log F(c^N z) + M \log z + \log A.$$

This implies that

$$\frac{F'(z)}{F(z)} = c^N \frac{F'(c^N z)}{F(c^N z)} + \frac{M}{z}$$

and thus

$$\frac{zF'(z)}{F(z)} - \frac{c^N z F'(c^N z)}{F(c^N z)} = M.$$

We obtain

$$\frac{r_2 e^{i\theta} F'(r_2 e^{i\theta})}{F(r_2 e^{i\theta})} - \frac{r_1 e^{i(\theta+\varphi)} F'(r_1 e^{i(\theta+\varphi)})}{F(r_1 e^{i(\theta+\varphi)})} = M,$$

where φ denotes the argument of c^N. Integrating the last equation with respect to θ completes the proof that the number of zeros of F in the annulus $r_1 < |z| < r_2$

equals M. We denote these zeros by $z_1, z_2, \ldots, z_M$. Then all zeros of F are given by $z_\mu c^{N\nu} = z_\mu q^{2\nu}$, where $\mu \in \{1, \ldots, M\}$ and $\nu \in \mathbb{Z}$. Hence

$$G(z) := \frac{F(z)}{\prod_{\mu=1}^{M} \theta\left(-\frac{z}{qz_\mu}, q\right)} \tag{4.8}$$

is holomorphic and has no zeros in $\mathbb{C} \setminus \{0\}$.

We deduce from (4.4) that

$$\theta(z, q) = qz\theta(q^2 z, q). \tag{4.9}$$

Using (4.3) and (4.9) we obtain

$$\begin{aligned} \frac{G(q^2 z)}{G(z)} &= \frac{F(q^2 z)}{F(z)} \prod_{\mu=1}^{M} \frac{\theta\left(-\frac{z}{qz_\mu}, q\right)}{\theta\left(-\frac{qz}{z_\mu}, q\right)} \\ &= \frac{1}{Az^M} \prod_{\mu=1}^{M} \left(-\frac{z}{z_\mu}\right) \\ &= \frac{(-1)^M}{A \prod_{\mu=1}^{M} z_\mu} \\ &=: B. \end{aligned} \tag{4.10}$$

Hence

$$q^2 z \frac{G'(q^2 z)}{G(q^2 z)} - z\frac{G'(z)}{G(z)} = 0.$$

We now define

$$H(z) := z\frac{G'(z)}{G(z)}$$

and deduce that H is holomorphic in $\mathbb{C} \setminus \{0\}$ and satisfies

$$H(q^2 z) = H(z). \tag{4.11}$$

Also if

$$K := \sup_{|q|^2 \le |z| \le 1} |H(z)|,$$

it follows from (4.11) that $|H(z)| \le K$ for $|q|^{2\nu} \le |z| \le |q|^{2\nu-2}$, $\nu \in \mathbb{Z}$, and so $H(z)$ is bounded in $\mathbb{C} \setminus \{0\}$. Thus H has removable singularities at 0 and ∞ and so is constant, say $H(z) = \alpha$. Integrating we obtain

$$G(z) = Cz^\alpha, \tag{4.12}$$

where $C \in \mathbb{C} \setminus \{0\}$. Since G is single-valued, we deduce that $\alpha \in \mathbb{Z}$. Returning to (4.10) we obtain $B = q^{2\alpha}$. Thus

$$\prod_{\mu=1}^{M} z_\mu = \frac{(-1)^M}{Aq^{2\alpha}}. \tag{4.13}$$

Combining (4.8) and (4.12) we find that

$$F(z) = Cz^{\alpha} \prod_{\mu=1}^{M} \theta\left(-\frac{z}{qz_\mu}, q\right). \tag{4.14}$$

It follows from (4.4) or (4.9) that

$$\theta(z,q) = q^{\alpha^2} z^{-\alpha} \theta(q^{-2\alpha} z, q)$$

so that

$$\theta\left(-\frac{z}{qz_1}, q\right) = q^{\alpha^2} \left(-\frac{qz_1}{z}\right)^{\alpha} \theta\left(-\frac{z}{q^{2\alpha+1} z_1}, q\right).$$

Thus replacing z_1 by $z_1' := z_1 q^{2\alpha}$ while leaving $z_2, \dots, z_M$ unaltered we see that (4.14) takes the form (4.6) while (4.13) becomes (4.7). □

4.2. **The asymptotic behaviour of theta functions.** In order to estimate the function F appearing in Theorem 4 we shall use the following lemma.

Lemma 3. *Suppose that* $0 < |q| < 1$ *and that* $z \in \mathbb{C}$, $|z| > 1$. *Define* ν *in* $\mathbb{N}$ *and* τ *in* $[0, 2)$ *by*

$$r := |z| = |q|^{2-\tau-2\nu}. \tag{4.15}$$

Then we have, uniformly as $z \to \infty$,

$$\log|\theta(z,q)| = \frac{(\log r)^2}{-4\log|q|} + \log|1 + q^{2\nu-1} z| + \mathcal{O}(1).$$

The lemma says that $\theta(z,q)$ is uniformly large except in small neighborhoods of its zeros $-q^{1-2\nu}$.

Proof. We use the product expansion (4.5). With z and ν as in the hypothesis we write

$$\log|\theta(z,q)| = \log|\omega| + \log\left|1 + q^{2\nu-1} z\right| + S_1 + S_2 + S_3$$

with

$$S_1 = \sum_{n=1}^{\infty} \log\left|1 + \frac{q^{2n-1}}{z}\right|$$

$$S_2 = \sum_{n=1}^{\nu-1} \log\left|1 + q^{2n-1} z\right|$$

and

$$S_3 = \sum_{n=\nu+1}^{\infty} \log\left|1 + q^{2n-1} z\right|.$$

Since $|q^{2n-1}| < 1 < |z|$ by hypothesis, we have

$$\left|\log\left|1 + \frac{q^{2n-1}}{z}\right|\right| < \log\frac{1}{1 - \left|\frac{q^{2n-1}}{z}\right|} < \log\frac{1}{1 - |q|^{2n-1}} < \frac{|q|^{2n-1}}{1 - |q|^{2n-1}} \le \frac{|q|^{2n-1}}{1 - |q|}$$

for $n \geq 1$. Thus

$$|S_1| < \sum_{n=1}^{\infty} \frac{|q|^{2n-1}}{1-|q|} = \frac{|q|}{(1-|q|)(1-|q|^2)}.$$

To estimate S_2 we note that

$$\begin{aligned} S_2 &= \sum_{n=1}^{\nu-1} \log\left|q^{2n-1}z\right| + \sum_{n=1}^{\nu-1} \log\left|1+\frac{q^{1-2n}}{z}\right| \\ &= (\nu-1)\log r + (\nu-1)^2 \log|q| + \sum_{n=1}^{\nu-1} \log\left|1+\frac{q^{1-2n}}{z}\right|. \end{aligned}$$

Also $|q^{1-2n}/z| < |q|^{1-2n+2\nu-2} = |q|^{2(\nu-n)-1} \leq |q| < 1$ for $1 \leq n \leq \nu-1$ so that

$$\left|\log\left|1+\frac{q^{1-2n}}{z}\right|\right| < \log\frac{1}{1-|q|^{2(\nu-n)-1}} < \frac{|q|^{2(\nu-n)-1}}{1-|q|^{2(\nu-n)-1}} \leq \frac{|q|^{2(\nu-n)-1}}{1-|q|}.$$

Thus

$$\sum_{n=1}^{\nu-1}\left|\log\left|1+\frac{q^{1-2n}}{z}\right|\right| < \sum_{n=1}^{\nu-1}\frac{|q|^{2(\nu-n)-1}}{1-|q|} < \sum_{j=1}^{\infty}\frac{|q|^{2j-1}}{1-|q|} = \frac{|q|}{(1-|q|)(1-|q|^2)}.$$

We obtain

$$\left|S_2 - \big((\nu-1)\log r + (\nu-1)^2\log|q|\big)\right| < \frac{|q|}{(1-|q|)(1-|q|^2)}.$$

For $n \geq \nu+1$ we have $|q^{2n-1}z| < |q|^{2n-1-2\nu} = |q|^{2(n-\nu)-1} \leq |q|$, and similarly as before we obtain

$$|S_3| < \sum_{n=\nu+1}^{\infty}\frac{|q|^{2(n-\nu)-1}}{1-|q|} = \frac{|q|}{(1-|q|)(1-|q|^2)}.$$

From our estimates for S_1, S_2, S_3 we deduce that

$$\log\left|\frac{\theta(z,q)}{1+q^{2\nu-1}z}\right| = (\nu-1)\log r + (\nu-1)^2\log|q| + \mathcal{O}(1). \tag{4.16}$$

From (4.15) we obtain

$$\nu - 1 = \frac{\log r}{-2\log|q|} - \frac{1}{2}\tau.$$

Combining this with (4.16) yields

$$\begin{aligned} \log\left|\frac{\theta(z,q)}{1+q^{2\nu-1}z}\right| &= \frac{(\log r)^2}{-2\log|q|} - \frac{1}{2}\tau\log r + \frac{1}{4}\left(\frac{\log r}{-\log|q|} - \tau\right)^2 \log|q| + \mathcal{O}(1) \\ &= \frac{(\log r)^2}{-4\log|q|} + \mathcal{O}(1). \end{aligned}$$

This proves Lemma 3. □

4.3. **Comparison of F and f.** With the hypotheses of Theorem 2 we write

$$f(z) = F(z) + R(z)$$

where F is defined by (4.1) and

$$R(z) = \sum_{n=-\infty}^{\infty} \delta_n z^n.$$

Comparing coefficients yields $\alpha_n = \eta_{r_n} \rho^{Nn^2} b^n + \delta_n$. From (1.7) we obtain

$$\delta_n = \mathcal{O}\left(\rho^{Nn^2} b^n \rho^{2n}\right) = \mathcal{O}\left(\rho^{Nn^2} (b\rho^2)^n\right)$$

as $n \to \infty$ while $\delta_n = -\eta_{r_n} \rho^{Nn^2} b^n$ for $n < 0$. This implies that

$$|R(z)| \le \sum_{n=1}^{\infty} |\delta_n||z|^n + \mathcal{O}(1) = \mathcal{O}\left(\theta(|b\rho^2 z|, |\rho|^N)\right)$$

and hence that

$$\log |R(z)| \le \log \theta(|b\rho^2 z|, |\rho|^N) + \mathcal{O}(1)$$

as $z \to \infty$. We deduce from Lemma 3 that if $r := |z|$ is large enough, then

$$\begin{aligned} \log |R(z)| &\le \frac{(\log r + \log |b\rho^2|)^2}{-4N \log |\rho|} + \mathcal{O}(1) \\ &\le \frac{1}{-4N \log |\rho|} (\log r)^2 + \frac{\log |b\rho^2|}{-2N \log |\rho|} \log r + \mathcal{O}(1). \end{aligned}$$

For μ in $\{1, 2, \ldots, M\}$ we choose the integer ν_μ according to Lemma 3; that is, we choose ν_μ such that $|q|^{3-2\nu_\mu} \le |z/z_\mu| < |q|^{1-2\nu_\mu}$. We also define $n_\mu := \nu_\mu - 1$. Combining Theorem 4 and Lemma 3 we find that

$$\begin{aligned} &\log |F(z)| \\ =\ & \sum_{\mu=1}^{M} \log \left| \theta\left(-\frac{z}{qz_\mu}, q\right) \right| + \log |C| \\ \ge\ & \sum_{\mu=1}^{M} \left(\frac{(\log r - \log |qz_\mu|)^2}{-4 \log |q|} + \log \left| 1 - \frac{q^{2\nu_\mu - 2} z}{z_\mu} \right| \right) - \mathcal{O}(1) \\ \ge\ & \sum_{\mu=1}^{M} \left(\frac{(\log r)^2}{-4 \log |q|} - \frac{\log |qz_\mu|}{-2 \log |q|} \log r + \log \left| 1 - \frac{q^{2n_\mu} z}{z_\mu} \right| \right) - \mathcal{O}(1) \\ =\ & \frac{M}{-4 \log |q|} (\log r)^2 + \left(\sum_{\mu=1}^{M} \frac{\log |qz_\mu|}{2 \log |q|} \right) \log r + \sum_{\mu=1}^{M} \log \left| 1 - \frac{q^{2n_\mu} z}{z_\mu} \right| - \mathcal{O}(1) \end{aligned}$$

Now $\rho = c^{1/(2M)}$ and $q^2 = c^N$ so that

$$NM \log |\rho| = \log |q|. \tag{4.17}$$

This implies that the coefficients of $(\log r)^2$ in the above estimates for $\log|R(z)|$ and $\log|F(z)|$ are equal. Thus

$$\log|R(z)| \leq \log|F(z)| - \sum_{\mu=1}^{M} \log\left|1 - \frac{q^{2n_\mu} z}{z_\mu}\right| - C_1 \log r + C_2, \tag{4.18}$$

where C_1 and C_2 are constants, with

$$C_1 = \frac{\log|b\rho^2|}{2N\log|\rho|} + \sum_{\mu=1}^{M} \frac{\log|qz_\mu|}{2\log|q|}.$$

Using (4.2), (4.7) and (4.17) we obtain

$$\begin{aligned} C_1 &= \frac{\log|b|}{2N\log|\rho|} + \frac{1}{N} + \sum_{\mu=1}^{M}\left(\frac{1}{2} + \frac{\log|z_\mu|}{2\log|q|}\right) \\ &= \frac{\log|b|}{2N\log|\rho|} + \frac{1}{N} + \frac{M}{2} + \frac{1}{2\log|q|}\log\left(\prod_{\mu=1}^{M}|z_\mu|\right) \\ &= \frac{\log|b|}{2N\log|\rho|} + \frac{1}{N} + \frac{M}{2} - \frac{\log|A|}{2\log|q|} \qquad (4.19) \\ &= \frac{\log|b|}{2N\log|\rho|} + \frac{1}{N} + \frac{M}{2} - \frac{\log\left|b^M\rho^{NM^2}\right|}{2NM\log|\rho|} \\ &= \frac{1}{N}. \end{aligned}$$

Suppose now that ζ is a zero of f. Then $|F(\zeta)| = |R(\zeta)|$ and we deduce from (4.18) and (4.19) that if $|\zeta|$ is sufficiently large, then

$$\sum_{\mu=1}^{M} \log\left|1 - \frac{q^{2n_\mu}\zeta}{z_\mu}\right| \leq -\frac{1}{N}\log|\zeta| + C_2. \tag{4.20}$$

This implies that there exists μ in $\{1, 2, \ldots, M\}$ with

$$\log\left|1 - \frac{q^{2n_\mu}\zeta}{z_\mu}\right| \leq -\frac{1}{NM}\log|\zeta| + \frac{C_2}{M}. \tag{4.21}$$

We shall obtain an inequality stronger than (4.21). In order to so, we denote by I_μ be the set of all λ in $\{1, 2, \ldots, M\}$ for which

$$\frac{q^{2n_\lambda}}{z_\lambda} = \frac{q^{2n_\mu}}{z_\mu}.$$

and by M_μ the cardinality of I_μ. Clearly $M_\mu \leq M$, and I_μ equals the set of all λ in $\{1, 2, \ldots, M\}$ for which there exists an integer ν such that $z_\lambda = q^{2\nu} z_\mu$. This implies that there exists a positive number δ such that if $\lambda \notin I_\mu$, then

$$\left|1 - \frac{q^{2n_\lambda} z_\mu}{q^{2n_\mu} z_\lambda}\right| = \left|1 - q^{2(n_\lambda - n_\mu)}\frac{z_\mu}{z_\lambda}\right| \geq \delta.$$

We may assume that $\delta < \frac{1}{2}$. Because of (4.21) we have

$$\left|1 - \frac{q^{2n_\mu}\zeta}{z_\mu}\right| < \frac{\delta}{4}$$

if ζ is large enough. Using the inequality $|1-\alpha\beta| \geq |1-\beta|(1-|1-\alpha|) - |1-\alpha|$ for $\alpha = q^{2n_\mu}\zeta/z_\mu$ and $\beta = q^{2n_\lambda}z_\mu/q^{2n_\mu}z_\lambda$ we obtain

$$\left|1 - \frac{q^{2n_\lambda}\zeta}{z_\lambda}\right| = \left|1 - \frac{q^{2n_\mu}\zeta}{z_\mu}\frac{q^{2n_\lambda}z_\mu}{q^{2n_\mu}z_\lambda}\right| \geq \delta\left(1 - \frac{\delta}{4}\right) - \frac{\delta}{4} \geq \frac{\delta}{2}.$$

Thus

$$\log\left|1 - \frac{q^{2n_\lambda}\zeta}{z_\lambda}\right| \geq \log\frac{\delta}{2}$$

for $\lambda \notin I_\mu$. We conclude that

$$\begin{aligned}\sum_{\lambda=1}^{M}\log\left|1 - \frac{q^{2n_\lambda}\zeta}{z_\lambda}\right| &= \sum_{\lambda\in I_\mu}\log\left|1 - \frac{q^{2n_\lambda}\zeta}{z_\lambda}\right| + \sum_{\lambda\notin I_\mu}\log\left|1 - \frac{q^{2n_\lambda}\zeta}{z_\lambda}\right| \\ &\geq M_\mu\log\left|1 - \frac{q^{2n_\mu}\zeta}{z_\mu}\right| + (M - M_\mu)\log\frac{\delta}{2}.\end{aligned}$$

Combining this with (4.20) we see that (4.21) can be improved to

$$\log\left|1 - \frac{q^{2n_\mu}\zeta}{z_\mu}\right| \leq -\frac{1}{NM_\mu}\log|\zeta| + C_3 \tag{4.22}$$

for some constant C_3. We deduce from (4.22) that

$$\left|1 - \frac{q^{2n_\mu}\zeta}{z_\mu}\right| = \mathcal{O}\left(|\zeta|^{-1/NM_\mu}\right).$$

This implies that $\zeta \sim z_\mu q^{-2n_\mu}$. More precisely, we have

$$\zeta = z_\mu q^{-2n_\mu}\left(1 + \mathcal{O}\left(q^{2n_\mu/NM_\mu}\right)\right) = z_\mu c^{-Nn_\mu}\left(1 + \mathcal{O}\left(c^{n_\mu/M_\mu}\right)\right). \tag{4.23}$$

On the other hand, an application of Rouché's theorem shows that if ε is fixed, small and positive and ν is sufficiently large, then $f = F + R$ and F have the same number of zeros in the disk $\{z \in \mathbb{C} : |z - q^{-2\nu}z_\mu| < \varepsilon\,|q^{-2\nu}|\}$. As F has a zero of multiplicity M_μ at $z = q^{-2\nu}z_\mu$ there, but no other zeros, we deduce that f has exactly M_μ zeros there, and each such zero ζ satisfies (4.23) as $\nu \to \infty$. This implies (1.8) and completes the proof of Theorem 2. ☐

Remark. It follows from the proof, together with (3.5), (4.2) and (4.7), that there exists an integer λ such that

$$\begin{aligned}\prod_{\mu=1}^{M} A_\mu &= q^{2\lambda}\prod_{\mu=1}^{M} z_\mu \\ &= (-1)^M q^{2\lambda}\frac{1}{A} \\ &= (-1)^M q^{2\lambda} b^{-M}\rho^{-NM^2} \\ &= (-1)^{M+1} q^{2\lambda}\frac{b_{j_{k-1},\ell}}{b_{j_k,\ell+M}}\rho^{M^2(j_{k-1}+j_k)}\rho^{-NM^2} \\ &= (-1)^{M+1} c^{N\lambda}\frac{b_{j_{k-1},\ell}}{b_{j_k,\ell+M}}c^{Mj_{k-1}}.\end{aligned}$$

Thus apart from the obvious factor $q^{2\lambda}=c^{N\lambda}$, we can express $\prod_{\mu=1}^{M} A_\mu$ in terms of the leading coefficients of the polynomials a_{j_k} and $a_{j_{k-1}}$. Recall that these are the polynomials marking the endpoints of the segment on the boundary of the Newton-Puiseux diagram whose slope σ_k yields the growth of f via (1.5).

5. Proof of Theorem 3

We begin with the following result.

Theorem 5. *Assume that* $0<q<1$, $\varphi\in\mathbb{R}$, $0<\eta<1$ *and* η *is irrational. Define*

$$\Phi(z) := \frac{\theta(ze^{i\eta\pi},q)}{\theta(ze^{-i\eta\pi},q)}$$

Suppose that $|A|\neq 0,1$ *and that*

$$\liminf_{r\to\infty}|\Phi(re^{i\varphi})| < |A| < \limsup_{r\to\infty}|\Phi(re^{i\varphi})|. \tag{5.1}$$

Then there exists a sequence (z_n) *such that* $z_n\to\infty$, $\Phi(z_n)=A$ *and* $\arg z_n\to\varphi$.

Corollary. *With* q,η,A *as above, the arguments of the zeros of*

$$\theta(ze^{i\eta\pi},q)-A\theta(ze^{-i\eta\pi},q)$$

are dense in some interval contained in $[-\pi,\pi]$, *but they are not dense in* $[-\pi,\pi]$.

Proof of Theorem 5. It follows from (4.9) that

$$\Phi(z)=e^{2\pi i\eta}\Phi(q^2z). \tag{5.2}$$

The hypothesis (5.1) ensures that there exists $z_0=r_0e^{i\varphi}$ such that $\Phi(z_0)=Ae^{i\lambda}$ where λ is real. If $\varepsilon>0$, there exists δ depending on ε such that $\Phi(z)$ assumes all values Ae^{it}, where $\lambda-\delta<t<\lambda+\delta$, in $|z-z_0|<\varepsilon|z_0|$. Writing $z_n:=q^{-2n}z_0$, we deduce from (5.2) that $\Phi(z)$ assumes all values Ae^{it}, where

$$\lambda-2\pi n\eta-\delta<t<\lambda-2\pi n\eta+\delta,$$

in $|z - z_n| < \varepsilon |z_n|$. Since η is irrational, infinitely many of the intervals

$$(\lambda - 2\pi n\eta - \delta, \lambda - 2\pi n\eta + \delta)$$

contain an integral multiple of 2π, and thus there are infinitely many values of n for which there exists z_n' satisfying $|z_n' - z_n| < \varepsilon |z_n|$ and $\Phi(z_n') = A$. Since ε can be chosen arbitrarily small, the conclusion follows. □

Proof of the Corollary. We shall show that (5.1) holds for some interval of values φ. We see that $|\theta(re^{i\varphi}, q)|$ decreases with $|\varphi|$ for $0 < |\varphi| < \pi$, when r is fixed. This implies that $|\Phi(z)| < 1$ if $0 < \arg z < \pi$ and $|\Phi(z)| > 1$ if $-\pi < \arg z < 0$, while $|\Phi(z)| = 1$ if z is real. Also, $|\Phi(z)|$ is not constant on any ray other than the real axis, because otherwise $u(z) := \log|\Phi(z)| - \mu \arg z$ for a suitable μ would vanish on the boundary of a sector not containing any zeros or poles, and thus, since u is bounded in such a sector by (5.2), u would vanish identically. Thus we can find φ satisfying (5.1), where $\varphi \in (0, \pi)$ if $|A| < 1$ and $\varphi \in (-\pi, 0)$ if $|A| > 1$. The set of such φ is open since, by (5.2),

$$\liminf_{r\to\infty} |\Phi(re^{i\varphi})| = \min_{q^2 \le r \le 1} |\Phi(re^{i\varphi})| \quad \text{and} \quad \limsup_{r\to\infty} |\Phi(re^{i\varphi})| = \max_{q^2 \le r \le 1} |\Phi(re^{i\varphi})|$$

which implies that the limes inferior and limes superior occuring in (5.1) are continuous in φ. Thus the set of all φ satisfying (5.1) contains an interval.

On the other hand, since $|\Phi(z)| < 1$ if $0 < \arg z < \pi$ while $|\Phi(z)| > 1$ if $-\pi < \arg z < 0$, the A-points of Φ and thus the zeros of $\theta(ze^{i\eta\pi}, q) - A\theta(ze^{-i\eta\pi}, q)$ are contained in a halfplane, and thus their arguments are not dense in $[-\pi, \pi]$. This proves the corollary. □

Proof of Theorem 3. We may assume that $c_1 = 1$ and define $A := -c_2$ and

$$F(z) := \theta(ze^{i\eta\pi}, \rho^2) - A\theta(ze^{-i\eta\pi}, \rho^2).$$

Then $f(z) = F(z) + \mathcal{O}(1/|z|)$ as $|z| \to \infty$. By the Corollary, there exists an interval I contained in $[-\pi, \pi]$ such that the arguments of the zeros of F are dense in I. We shall show that the arguments of the zeros of f are also dense in I. We assume that this is not the case. Then there exists φ_0 in I and a positive number δ such that $[\varphi_0 - 2\delta, \varphi_0 + 2\delta] \subset I$ and $\{re^{i\varphi} : |\varphi - \varphi_0| < 2\delta\}$ contains no zero of f. Now

$$\log M(r, f) \sim \frac{1}{-2\log c} (\log r)^2 = \frac{1}{-8\log \rho} (\log r)^2$$

by (1.5). It follows from a theorem of the second author [5, Theorem 2] that

$$\log|f(z)| \sim \log M(r, f) \sim \frac{1}{-8\log\rho} (\log r)^2 \tag{5.3}$$

uniformly as $z = re^{i\varphi} \to \infty$ outside an $\mathcal{E}$-set, i. e. a set of disks E_ν subtending angles at the origin, whose sum is finite. It also follows from the construction of the E_ν (see [6, Lemma 1] and [5, Lemma 4, p. 482]) that each E_ν contains at least one zero of f. Thus for large ν the E_ν do not meet the sector

$$S := \left\{re^{i\varphi} : |\varphi - \varphi_0| < \delta\right\}.$$

Hence (5.3) holds uniformly as $z \to \infty$ in S. But Theorem 5 implies that S contains a sequence (z_n) of zeros of f, and $z_n \to \infty$ with n. Hence

$$f(z_n) = F(z_n) + o(1) = o(1)$$

as $n \to \infty$. This contradicts (5.3), so that the arguments of the zeros f must be dense in I.

On the other hand, the Corollary says that the arguments of the zeros of F are not dense in $[-\pi, \pi]$. An argument similar to the one above now shows that the arguments of the zeros of f are also not dense in $[-\pi, \pi]$. □

6. Further examples

Example 2. Suppose that $N \geq 2$ and let f be defined by (1.6) with

$$\alpha_n := \rho^{Nn^2}(1 - \rho^{2n}).$$

Define $c := \rho^2$. Then f satisfies the equation

$$f(z) - \frac{1}{\rho^2} f(cz) - \rho^N z f(c^N z) + \rho^{N+2} z f(c^{N+1} z) = 0. \tag{6.1}$$

The vertices of the Newton-Puiseux diagram are $(0,0)$ and $(N,1)$. The slope σ_1 of the segment connecting these points is given by $\sigma_1 = 1/N$. We find that f satisfies (1.5) with $k = 1$ and $\sigma_1 = 1/N$. Of course, this also follows directly from Lemma 1. Note, however, that the above argument also works for any other entire solution f of (6.1).

The example shows that the error term in (1.7) cannot be improved in the case that $M = 1$. Considering $f(z^M)$ instead of $f(z)$ we find that the error term in (1.7) is also best possible if $M > 1$.

With $q := \rho^N$ we have

$$f(z) = \theta(z, q) - \theta(\rho^2 z, q) + \mathcal{O}\left(\frac{1}{|z|}\right)$$

as $z \to \infty$. We deduce that if $f(z) = 0$, then

$$\theta(z, q) = \theta(\rho^2 z, q) + \mathcal{O}\left(\frac{1}{|z|}\right).$$

Lemma 3 implies, with $r := |z|$ and ν chosen as in the hypothesis of Lemma 3, that

$$\frac{(\log r)^2}{-4 \log |q|} + \log |1 + q^{2\nu - 1} z| = \frac{(\log r + 2 \log |\rho|)^2}{-4 \log |q|} + \log |1 + q^{2\mu - 1} \rho^2 z| + \mathcal{O}(1),$$

where $\mu = \nu$ or $\mu = \nu - 1$. As in §4.3 we obtain

$$\begin{aligned} \log |1 + q^{2\nu-1} z| &= \frac{\log |\rho|}{-\log |q|} \log r + \log |1 + q^{2\mu-1} \rho^2 z| + \mathcal{O}(1) \\ &= -\frac{1}{N} \log r + \log |1 + q^{2\mu-1} \rho^2 z| + \mathcal{O}(1). \end{aligned} \tag{6.2}$$

This implies, as we know already from Theorem 2 and the remark at the end of §4.3, that for sufficiently large n in $\mathbb{N}$ there exists a zero z_n of f satisfying $z_n \sim -q^{1-2n}$ as $n \to \infty$, say $z_n = -q^{1-2n}(1+\varepsilon_n)$ where $\varepsilon_n \to 0$. Substituting this in (6.2) yields

$$\log|\varepsilon_n| = -\frac{1}{N}\log|z_n| + \mathcal{O}(1) = \frac{2n}{N}\log|q| + \mathcal{O}(1) = 2n\log|\rho| + \mathcal{O}(1).$$

Hence $\log|\varepsilon_n \rho^{-2n}| = \mathcal{O}(1)$ so that $\varepsilon_n \neq o(\rho^{2n})$. This shows that the error term in (1.9) is best possible.

Example 3. Let γ, $t_{1,2}$, ρ and c be as in Example 1. For k in $\{1,2\}$ the function

$$f_k(z) := \prod_{j=0}^{\infty}\left(1 - \frac{c^j\rho^2 z}{t_k}\right)$$

is a solution of

$$f(z) + \left(2\gamma\rho^2 z - 1 - \frac{1}{c}\right) f(cz) + \left(c^2z^2 - 2\gamma\rho^2 z + \frac{1}{c}\right) f(c^2 z) = 0.$$

The Newton-Puiseux diagram is the same as in Example 1. Again the hypothesis of Theorem 2 is not satisfied.

We note that (4.5) implies that

$$\omega f_{1,2}(z) = \left(1 + \mathcal{O}\left(\frac{1}{|z|}\right)\right)\theta(ze^{\mp i\eta\pi}, \rho^2)$$

with $\omega = \prod_{n=1}^{\infty}(1-\rho^{4n})$. The argument used in §5 now shows that the conclusion of Theorem 3 also holds for $f = c_1 f_1 + c_2 f_2$.

Example 4. Suppose that $\gamma \in \mathbb{C}\backslash\{0\}$ and let $t_{1,2}$ be the solutions of $\gamma t^2 + t - 1 = 0$. We consider (1.1) with $m = 2$, $Q(z) \equiv 0$, $a_0(z) \equiv 1$,

$$a_1(z) := \gamma z^2 + \left(1 + \frac{1}{c}\right) z - 1 - \frac{1}{c^2}$$

and

$$a_2(z) := \gamma c z^3 + (1-\gamma)z^2 - \frac{2}{c}z + \frac{1}{c^2}.$$

A computation shows that two solutions are given by

$$f_1(z) := \prod_{\mu=1}^{2}\prod_{j=0}^{\infty}\left(1 - \frac{c^j z}{t_\mu}\right)$$

and

$$f_2(z) := \prod_{j=0}^{\infty}\left(1 - c^j z\right).$$

The vertices of the Newton-Puiseux diagram are $(0,0)$, $(1,2)$ and $(2,3)$, and we have $\sigma_1 = 2$ and $\sigma_2 = 1$. We find that f_1 satisfies (1.5) with $k = 1$ and f_2 satisfies (1.5) with $k = 2$.

In the special case that $\gamma = -\frac{1}{4}$ we have $t_1 = t_2 = 2$ and thus

$$f_1(z) = \left(\prod_{j=0}^{\infty} \left(1 - \frac{c^j z}{2} \right) \right)^2 .$$

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