On the Derivative of Meromorphic Functions with Multiple Zeros*

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Let f be a transcendental meromorphic function and let R be a rational function, $R \not\equiv 0$. We show that if all zeros and poles of f are multiple, except possibly finitely many, then $f' - R$ has infinitely many zeros. If f has finite order and R is a polynomial, then the conclusion holds without the hypothesis that poles be multiple.

1. INTRODUCTION AND RESULTS

Let F be a transcendental meromorphic function, $c \in \mathbb{C}\backslash\{0\}$ and $n \in \mathbb{N}$. (Here and in the following, unless stated otherwise, "meromorphic" always means "meromorphic in the complex plane $\mathbb{C}$.") Hayman [5, Corollary to Theorem 9] proved that if $n \geq 3$, then $F'F^n - c$ has infinitely many zeros. He conjectured that this also holds for $n = 1$ and $n = 2$. This conjecture was confirmed by Mues [12, Satz 3] for the case $n = 2$ and finally the case $n = 1$ was settled in [2, 3, 20]. Actually the method of [2, 3, 20] applies for all $n \in \mathbb{N}$.

The structure of the proof of Hayman's conjecture in [2, 3, 20] is as follows. First it was proved in [2] that the conjecture is true for functions

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of finite order. Then normal family arguments (cf. Lemma 2.1 below) were used to reduce the general case to the finite order case.

We note that if $n \in \mathbb{N}$ and $f := \frac{1}{n+1}F^{n+1}$, then $f' = F'F^n$, and f has only multiple zeros and poles. It turns out that some results concerning functions of the form $F'F^n$ hold more generally for derivatives of functions with multiple zeros.

In the case of finite order we have the following results.

Theorem A. [2, Theorem 3]. *Let f be a meromorphic function of finite order and $c \in \mathbb{C}\backslash\{0\}$. If f has infinitely many multiple zeros, then $f' - c$ has infinitely many zeros.*

Theorem B. [18, Lemma 6]. *Let f be a transcendental meromorphic function of finite order and $c \in \mathbb{C}\backslash\{0\}$. If f has only multiple zeros, then $f' - c$ has infinitely many zeros.*

While it was shown in [2, p. 370] that the hypothesis that f be of finite order cannot be omitted in Theorem A, we do not know whether it is necessary in Theorem B.

For functions of unrestricted growth, we have the following result.

Theorem C. [18, Theorem 1]. *Let f be a transcendental meromorphic function and $c \in \mathbb{C}\backslash\{0\}$. If f has only multiple zeros and poles, then $f' - c$ has infinitely many zeros.*

A discussion of the case where f is rational leads to the following result.

Theorem D. (cf. [18, Lemma 9]). *Let f be a meromorphic function and $c \in \mathbb{C}\backslash\{0\}$. If f has only multiple zeros and poles, and $f' - c$ has no zeros, then f is constant.*

While it is not known whether the hypothesis that poles be multiple is necessary in Theorem C, it cannot be omitted in Theorem D, as shown by the example $f(z) = c(z-1)^2/z$.

The question whether the constant c in the above results can be replaced by a rational function was addressed in [1]. It was shown in [1] that if F is a transcendental meromorphic function of finite order and P is a polynomial which does not vanish identically, then $F'F - P$ has infinitely many zeros. The method used also shows that $F'F^n - P$ has infinitely many zeros for every $n \in \mathbb{N}$. Here we remove the restriction on the order and also allow a rational function instead of a polynomial.

Theorem 1.1. *Let f be a transcendental meromorphic function and let R be a rational function, $R \not\equiv 0$. Suppose that all zeros and poles of f are multiple, except possibly finitely many. Then $f' - R$ has infinitely many zeros.*

It seems reasonable to conjecture that the conclusion of Theorem 1.1 holds without the hypothesis that the poles be multiple. In this direction, we have the following result.

THEOREM 1.2. *Let f be a transcendental meromorphic function of finite order and let P be a polynomial, $P \not\equiv 0$. Suppose that all zeros of f are multiple, except possibly finitely many. Then $f' - P$ has infinitely many zeros.*

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2. PROOF OF THEOREM 1.1

The main tool in the proof of Theorem 1.1 is the following result.

LEMMA 2.1. *Let $\mathcal{F}$ be a family of functions meromorphic in a domain $D \subset \mathbb{C}$ and let $m \in \mathbb{N}$ and $\alpha \in \mathbb{R}$ with $-m < \alpha < 1$. Suppose that the zeros of the functions in $\mathcal{F}$ have multiplicity at least m and that $\mathcal{F}$ is not normal at $z_0 \in D$. Then there exist a sequence (f_k) in $\mathcal{F}$, a sequence (z_k) in D, a sequence (ρ_k) of positive real numbers and a non-constant function f which is meromorphic in $\mathbb{C}$ such that $z_k \to z_0$, $\rho_k \to 0$ and*

$$\rho_k^\alpha f_k(z_k + \rho_k z) \to f(z)$$

locally uniformly in $\mathbb{C}$.

Moreover, the spherical derivative $f^\# := |f'|/(1 + |f|^2)$ of f satisfies $f^\#(z) \le f^\#(0) = 1$ for all $z \in \mathbb{C}$. In particular, f has finite order.

The case $\alpha = 0$ of this lemma is due to Zalcman [19]. Pang [14, 15] proved that one can always take $-1 < \alpha < 1$. Pang and Xue [16] showed that $\alpha < 0$ is admissible if the functions in $\mathcal{F}$ have no zeros. The above version is due to Chen and Gu [4, Theorem 2]. For a survey of applications of this lemma we refer to [21].

Next we recall that a meromorphic function g is called a *Julia exceptional function* if $g^\#(z) = O(1/|z|)$ as $|z| \to \infty$.

LEMMA 2.2. *Let g be a meromorphic function which is not a Julia exceptional function. Then there exists a sequence (a_k) in $\mathbb{C}$ such that $a_k \to \infty$, $a_k g'(a_k) \to \infty$ and $g(a_k) \to 0$ as $k \to \infty$.*

Proof. Since g is not a Julia exceptional function, there exists a sequence (b_n) in $\mathbb{C}$ such that $b_n \to \infty$ and $b_n g^\#(b_n) \to \infty$. We define $g_n(z) := g(b_n z)$. Then $g_n^\#(1) = |b_n| g^\#(b_n) \to \infty$ so that (g_n) is not normal at 1. Using Lemma 2.1 for $\alpha = 0$ we obtain sequences (n_k), (z_k) and (ρ_k) satisfying $n_k \in \mathbb{N}$, $n_k \to \infty$, $z_k \in \mathbb{C}$, $z_k \to 1$, $\rho_k > 0$ and $\rho_k \to 0$ such that

$$g_{n_k}(z_k + \rho_k z) \to h(z)$$

for some non-constant function h meromorphic in $\mathbb{C}$. Given $\varepsilon > 0$, there exists $\xi \in \mathbb{C}$ with $|h(\xi)| < \varepsilon$ and $h'(\xi) \neq 0$. With $c_k := (z_k + \rho_k \xi) b_{n_k}$ we have

$$g(c_k) = g_{n_k}(z_k + \rho_k \xi) \to h(\xi)$$

as $k \to \infty$. Moreover,

$$\rho_k b_{n_k} g'(c_k) = \rho_k g'_{n_k}(z_k + \rho_k \xi) \to h'(\xi)$$

as $k \to \infty$. Since $\rho_k \to 0$ and $z_k \to 1$ we have $c_k \sim b_{n_k}$ as $k \to \infty$. This yields

$$c_k g'(c_k) = (1 + o(1)) \frac{h'(\xi)}{\rho_k} \to \infty$$

as $k \to \infty$. Altogether we have thus found a sequence (c_k) in $\mathbb{C}$ such that $c_k \to \infty$, $c_k g'(c_k) \to \infty$ and $\limsup_{k \to \infty} |g(c_k)| \leq \varepsilon$. Since ε can be chosen arbitrarily small, we deduce that a sequence (a_k) with the properties stated exists. ▮

Next we need the following result of Lehto and Virtanen [11, p. 7].

Lemma 2.3. *A transcendental Julia exceptional function does not have an asymptotic value.*

We shall use some standard terminology and results from Nevanlinna theory; see [7, 10, 13]. It follows easily from the Ahlfors-Shimizu form of the Nevanlinna characteristic that if f is a Julia exceptional function, then $T(r, f) = O((\log r)^2)$ as $r \to \infty$.

We shall need the following two results concerning functions satisfying this growth condition. The first one is due to Hayman [6, Corollary to Theorem 1]; see also his book [8, p. 442, Corollary 4].

Lemma 2.4. *Let h be an entire function satisfying*

$$\log M(r, h) = O((\log r)^2) \tag{1}$$

as $r \to \infty$. Then $\log |h(re^{i\theta})| \sim \log M(r, h)$ as $r \to \infty$ for almost every $\theta \in [0, 2\pi]$.

We shall use Lemma 2.4 to prove the following result.

Lemma 2.5. *Let f be a transcendental meromorphic function and let R be a rational function satisfying $R(z) \sim cz^d$ as $z \to \infty$, with $c \in \mathbb{C} \backslash \{0\}$ and $d \in \mathbb{Z}$. Suppose that $f' - R$ has only finitely many zeros and that $T(r, f) = O((\log r)^2)$ as $r \to \infty$. Define $g(z) := f(z)/z^{d+1}$, with $g := f$ if $d = -1$. Then g has an asymptotic value.*

Proof. Since $f' - R$ has only finitely many zeros, there exists a polynomial $P \not\equiv 0$ such that $h := P/(f' - R)$ is entire. By standard results in Nevanlinna theory, we have

$$\log M(r,h) \leq 3T(2r,h) \leq 3T(2r,f') + O(\log r)$$

and

$$T(r,f') \leq 2T(r,f) + m\left(r, \frac{f'}{f}\right) = 2T(r,f) + O(\log r)$$

as $r \to \infty$. Thus h satisfies (1). Let $m := \deg(P) + 2 + |d|$. By Lemma 2.4 there exists $\theta \in [0, 2\pi]$ such that $|h(re^{i\theta})|r^{-m} \to \infty$ as $r \to \infty$. Hence

$$|f'(re^{i\theta}) - R(re^{i\theta})| = \left|\frac{P(re^{i\theta})}{h(re^{i\theta})}\right| \leq \frac{1}{r^{2+|d|}}$$

for sufficiently large r, say $r \geq r_0$. It follows that

$$\int_{r_0}^{r} \left(f'(te^{i\theta}) - R(te^{i\theta})\right) dt$$

tends to a finite limit as $r \to \infty$. If $d \geq 0$ we obtain

$$f(re^{i\theta}) \sim \frac{c}{d+1}\left(re^{i\theta}\right)^{d+1}$$

as $r \to \infty$. Thus $g(re^{i\theta}) \to c/(d+1)$ as $r \to \infty$. If $d \leq -2$ we obtain

$$f(re^{i\theta}) = a + \frac{c}{d+1}\left(re^{i\theta}\right)^{d+1} + O\left(r^{d}\right)$$

for some $a \in \mathbb{C}$ as $r \to \infty$. If $a = 0$ we find again that $g(re^{i\theta}) \to c/(d+1)$ as $r \to \infty$, while $g(re^{i\theta}) \to \infty$ if $a \neq 0$. Finally, if $d = -1$, then $f(re^{i\theta}) \sim c \log r$ so that $g(re^{i\theta}) = f(re^{i\theta}) \to \infty$ as $r \to \infty$. ∎

Proof of Theorem 1.1. We assume that $f' - R$ has only finitely many zeros. We choose c, d and g as in Lemma 2.5; that is, $R(z) \sim cz^d$ as $z \to \infty$ and $g(z) = f(z)/z^{d+1}$. First we assume that g is a Julia exceptional function. Then $T(r,g) = O((\log r)^2)$ and hence $T(r,f) = O((\log r)^2)$ as $r \to \infty$, and thus g has an asymptotic value by Lemma 2.5. This is a contradiction to Lemma 2.3.

Thus g is not a Julia exceptional function and hence there exists a sequence (a_k) as in Lemma 2.2. We then have

$$\frac{f(a_k)}{a_k^{d+1}} = g(a_k) \to 0 \tag{2}$$

and

$$\frac{f'(a_k)}{a_k^d} = a_k g'(a_k) + (d+1)g(a_k) \to \infty \tag{3}$$

as $k \to \infty$.

First we consider the case that $d \neq -1$. For $D := \{z \in \mathbb{C} : |z-1| < \frac{1}{2}\}$ and $\mu := \frac{1}{d+1}$ we consider the function $h_k : D \to \widehat{\mathbb{C}} := \mathbb{C} \cup \{\infty\}$ defined by

$$h_k(z) = \frac{f(a_k z^\mu)}{c\mu a_k^{d+1}}.$$

Here z^μ denotes the branch of the root that fixes 1. We have

$$h_k'(z) = \frac{f'(a_k z^\mu) z^\mu}{c a_k^d z} \neq \frac{R(a_k z^\mu) z^\mu}{c a_k^d z}$$

if $z \in D$ and k is sufficiently large. With

$$s_k(z) := \frac{R(a_k z^\mu) z^\mu}{c a_k^d z}$$

we thus have

$$h_k'(z) \neq s_k(z) \tag{4}$$

if $z \in D$ and k is sufficiently large. By the definition of c, d and μ we have

$$s_k(z) \to 1 \tag{5}$$

as $k \to \infty$, uniformly for $z \in D$. By (2) and (3) we have

$$h_k(1) = \frac{f(a_k)}{c\mu a_k^{d+1}} \to 0$$

and

$$h_k'(1) = \frac{f'(a_k)}{c a_k^d} \to \infty$$

as $k \to \infty$. Thus $h_k^\#(1) \to \infty$ as $k \to \infty$. This implies that (h_k) is not normal at 1. For sufficiently large k all zeros and poles of h_k in D are multiple. Thus we can apply Lemma 2.1 with $\alpha = -1$ and obtain sequences $(k_j), (z_j)$ and (ρ_j) satisfying $k_j \in \mathbb{N}$, $k_j \to \infty$, $z_j \in D$, $z_j \to 1$, $\rho_j > 0$ and $\rho_j \to 0$ such that

$$\frac{h_{k_j}(z_j + \rho_j z)}{\rho_j} \to h(z)$$

for some non-constant function h meromorphic in $\mathbb{C}$. By Hurwitz's theorem, h has only multiple zeros and poles. Since $h'_{k_j}(z_j+\rho_j z)\to h'(z)$ we deduce from (4), (5) and Hurwitz's theorem that $h'-1$ has no zeros. This contradicts Theorem D.

We now consider the case that $d=-1$. Here we define $D:=\{z\in\mathbb{C}: |z|<1\}$ and $h_k: D\to\widehat{\mathbb{C}}$ by

$$h_k(z)=\frac{f(a_k e^z)}{c}.$$

Then

$$h'_k(z)=\frac{f'(a_k e^z)a_k e^z}{c}$$

and with

$$s_k(z)=\frac{R(a_k e^z)a_k e^z}{c}$$

we find again that (4) and (5) hold. Similarly as before we have $h_k^{\#}(0)\to\infty$, and an application of Lemma 2.1 leads again to a contradiction. ∎

3. PROOF OF THEOREM 1.2

We shall use arguments similar to those used in [1]. As in [1] we need the following result proved in [2, Corollary 3].

LEMMA 3.1. *Let g be a meromorphic function of finite order. If g has only finitely many critical values, then g has only finitely many asymptotic values.*

The next result is due to Rippon and Stallard [17, Lemma 2.2]

LEMMA 3.2. *Let g be a transcendental meromorphic function and suppose that the set of all finite critical and asymptotic values of g is bounded. Then there exists $R>0$ such that if $|z|>R$ and $|g(z)|>R$, then*

$$|g'(z)|\geq\frac{|g(z)|\log|g(z)|}{16\pi|z|}.$$

Finally we need the following lemma which follows from a result of Hua [9].

LEMMA 3.3. *Let f be a transcendental meromorphic function and let P be a polynomial, $P\not\equiv 0$. Then at least one of the function f and $f'-P$ has infinitely many zeros.*

This extends a classical result of Hayman (see [5, Theorem 3] or [7, Corollary to Theorem 3.5]) dealing with the case that P is constant.

Proof of Theorem 1.2. We choose a polynomial Q such that $Q' = P$ and define $g := f - Q$. We assume that $g' = f' - P$ has only finitely many zeros. Then g has only finitely many asymptotic values by Lemma 3.1, and thus g satisfies the hypotheses of Lemma 3.2. It follows from Lemma 3.3 that f has infinitely many zeros, say $f(z_k) = 0$, with $z_k \to \infty$ as $k \to \infty$. We clearly have $g(z_k) = -Q(z_k)$. Since f has only finitely many simple zeros, z_k is a multiple zero of f and hence $g'(z_k) = -Q'(z_k)$ for large k. Lemma 3.2 yields

$$|Q'(z_k)| = |g'(z_k)| \geq \frac{|g(z_k)| \log |g(z_k)|}{16\pi |z_k|} = \frac{|Q(z_k)| \log |Q(z_k)|}{16\pi |z_k|}$$

and thus

$$\frac{|z_k Q'(z_k)|}{|Q(z_k)|} \geq \frac{\log |Q(z_k)|}{16\pi}$$

for large k. This is a contradiction, since the left side of the last inequality tends to $\deg(Q)$ as $k \to \infty$, while the right side tends to ∞. ▮

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