Semihyperbolic entire functions

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Abstract. The concept of semihyperbolicity introduced by Carleson, Jones and Yoccoz for polynomials is carried over to transcendental entire functions. For certain classes of semihyperbolic entire functions it is shown that there are no wandering domains and that the Julia sets are locally connected.

1. Introduction

In complex dynamics the singularities of the inverse function play an important role. For example, they are closely related to periodic components of the Fatou set, cf., e.g., [19, §2.4]. For rational functions the set of singularities of the inverse consists precisely of the set of critical values; for entire functions we also have to consider the asymptotic values. We call the critical and asymptotic values *singular values*.

Already Fatou [13, §34] considered rational functions for which the ω-limit set of the critical values does not intersect the Julia set. Today such functions are called *hyperbolic*. A weakened form of hyperbolicity called *subhyperbolicity* was introduced by Douady and Hubbard [9, Exposé III]. Furthermore, building on work by Mañé [16], Carleson, Jones, and Yoccoz [8] introduced (for polynomials) the concept of *semihyperbolicity*. One aspect of the papers mentioned is to relate these concepts of hyperbolicity to the geometry of the Julia set. In particular, it is shown there that the above conditions imply that the Julia set is locally connected if it is connected.

In this paper, we consider semihyperbolic entire functions. We note that Kriete and Sumi [14] treated the more general case of semihyperbolic entire semigroups, but our results are in a somewhat different direction. We show that semihyperbolic entire functions do not have wandering domains in which the iterates have a finite limit function. This is used to show that certain entire functions do not have any wandering domains. We also show that the Julia sets of certain semihyperbolic transcendental entire functions are locally connected. We illustrate our results by a number of examples.

The readers are expected to be familiar with the basic notations and results of complex dynamics, which can be found in, e.g., [4, 7, 17, 21] for the dynamics of rational functions and [5, 11, 19] for those of transcendental entire functions.

2. Results

Let f be an entire function. We denote the n-th iterate of f by f^n, and the Fatou set and the Julia set of f by $F(f)$ and $J(f)$, respectively. For $a \in \mathbb{C}$ and $r > 0$, we use the notation $D(a,r) = \{z \in \mathbb{C} \mid |z-a| < r\}$. We say that f is *semihyperbolic at* a if there exist $r > 0$ and $N \in \mathbb{N}$ such that for all $n \in \mathbb{N}$ and for all components U of $f^{-n}(D(a,r)) = \{z \in \mathbb{C} \mid f^n(z) \in D(a,r)\}$ the function $f^n|_U : U \to D(a,r)$ is a proper map of degree at most N. We say that f is *semihyperbolic* if f is semihyperbolic at all $a \in J(f)$. Recall that the ω-limit set of a point a consists of all $b \in \mathbb{C}$ for which there exists an increasing sequence (n_k) such that $f^{n_k}(a) \to b$. A non-periodic point which is contained in its own ω-limit set is called *recurrent*. Mañé [16] showed that if f is rational and $a \in J(f)$ is not a parabolic periodic point and not in the ω-limit set of a recurrent critical point, then f is semihyperbolic at a. Conversely, it is easy to see that a rational or entire function is not semihyperbolic at a parabolic periodic point or a recurrent critical point. Furthermore, an entire function is never semihyperbolic at an asymptotic value. We note, however, that there exists a transcendental entire function which has no asymptotic value, no parabolic periodic point and no recurrent critical point, but which is not semihyperbolic; see Example 1 below.

In [8], semihyperbolic polynomials are characterized by various conditions. The condition given in the following theorem is one of them. It was also obtained by Kriete and Sumi [14] in the case of semihyperbolic transcendental semigroups. For $U \subset \mathbb{C}$ we denote by $\operatorname{diam}(U)$ the spherical diameter of U.

Theorem 1 *Let f be entire and suppose that f is semihyperbolic at $a \in J(f)$. Then there exists $s > 0$ with the following property: for all $\varepsilon > 0$ there exists $M \in \mathbb{N}$ such that if $n \geq M$ and U is a component of $f^{-n}(D(a,s))$, then* $\operatorname{diam}(U) < \varepsilon$.

For completeness we include a proof of Theorem 1 in §3 below.

Corollary 1 *Let f be entire. Assume that $F(f)$ has a Siegel disk U. Then f is not semihyperbolic at any point of ∂U.*

Another consequence of Theorem 1 is the following result.

Theorem 2 *Let f be entire. If f is semihyperbolic at $a \in \mathbb{C}$, then a is not a limit function of $\{f^n\}_{n\in\mathbb{N}}$ in any component of $F(f)$.*

Corollary 2 *A semihyperbolic entire function does not have a wandering domain where the iterates have a finite limit function.*

We remark that a semihyperbolic entire function may have wandering domains. For example, the function $z \mapsto z + e^{-z} - 1 + 2\pi i$ is semihyperbolic and it has a wandering domain where the iterates tend to ∞; see [1].

We denote the set of all singular values of f by $\text{sing}(f^{-1})$ and define $P(f) = \overline{\bigcup_{n=0}^{\infty} f^n(\text{sing}(f^{-1}))}$. It was shown in [6] that finite limit functions in wandering domains are contained in the derived set of $P(f)$. Using Theorem 2 and Corollary 2 we can prove the non-existence of wandering domains for some functions where the argument of [6] does not seem to apply; see Examples 2–4 below.

To exclude wandering domains in these examples, the main tool used besides Theorem 2 and Corollary 2 is a theorem of Eremenko and Lyubich [12, Theorem 1] which says that if $\text{sing}(f^{-1})$ is bounded, then there does not exist a component of $F(f)$ where the iterates of f tend to ∞. We denote the class of all transcendental entire functions f for which $\text{sing}(f^{-1})$ is bounded by $\mathcal{B}$.

Corollary 3 *If $f \in \mathcal{B}$ is semihyperbolic, and $F(f) \neq \emptyset$, then $F(f)$ consists only of attracting basins.*

Now we turn our attention to the local connectivity of Julia sets. Local connectivity of Julia sets of certain transcendental entire functions was considered in [2, 3, 18].

Theorem 3 *Let f be entire and U be a bounded invariant component of $F(f)$. Assume that, for every $a \in \partial U$, there exist $r > 0$ and $N \in \mathbb{N}$ such that for $n \in \mathbb{N}$ every component V of $f^{-n}(D(a,r))$ with $V \cap \partial U \neq \emptyset$ satisfies $\deg(f^n|_V : V \to D(a,r)) \leq N$. Then the boundary of U is a Jordan curve. In particular, if f is semihyperbolic on ∂U, then ∂U is a Jordan curve.*

For the Julia set, we have the following.

Theorem 4 *Let f be entire. Assume that $F(f)$ consists of finitely many attracting basins. Suppose that if U is an immediate attracting basin, then U is bounded, f is semihyperbolic on ∂U, and there exists $N \in \mathbb{N}$ such that for every $n \in \mathbb{N}$ and for every component $V \neq U$ of $f^{-n}(U) \setminus \bigcup_{k=0}^{n-1} f^{-k}(U)$ we have $\deg(f^n|_V : V \to U) \leq N$. Then $J(f)$ is locally connected.*

We note here that if an entire function f has an unbounded invariant component U, then ∂U and $J(f)$ are not locally connected, except possibly if $f|_U$ is univalent; cf. [2, 3].

3. Proofs of Theorem 1 and Theorem 2

Proof of Theorem 1. Let r and N be as in the definition of semihyperbolicity. We shall show that each $s \in (0,r)$ has the required property. To do this, we assume that there exists $s \in (0,r)$ for which this is false. We may assume that $a = 0$ and $r = 1$. Thus $0 < s < 1$, and there exist $\varepsilon > 0$, a sequence (n_k) tending to ∞ and components U_k of $f^{-n_k}(D(0,s))$ such that $\text{diam}(U_k) \geq \varepsilon$ for all k. This implies that there exist $R > 0$ with $D(0,R) \cap U_k \neq \emptyset$ for all k, say $u_k \in D(0,R) \cap U_k$. Let V_k be the component of $f^{-n_k}(\mathbb{D})$ that contains U_k. Then V_k is simply-connected, and thus there exists a biholomorphic map $\phi_k : \mathbb{D} \to V_k$ with $\phi_k(0) = u_k$. Then $B_k := f^{n_k} \circ \phi_k$ is a Blaschke product of degree at most N, with $|B_k(0)| = |f^{n_k}(u_k)| < s$. Next we note that there exists $b \in \mathbb{C}\backslash\mathbb{D}$ with $f^n(b) \notin \mathbb{D}$ for all $n \in \mathbb{N}$. (For example, this follows from the fact that $\{z \in \mathbb{C} \mid f^n(z) \to \infty\}$ is not empty; see [10].) It follows that $b \notin V_k$ for all k. Hence $|\phi_k'(0)| \leq 4|b - u_k| \leq 4(|b| + R)$ by Koebe's one quarter theorem. This implies that the ϕ_k form a normal family. Passing to subsequences if necessary, we may assume that $\phi_k \to \phi$ for some holomorphic function $\phi : \mathbb{D} \to \mathbb{C}$ and that $B_k \to B$ for some Blaschke product B satisfying $|B(0)| \leq s$. There exists $\rho \in (0, 1)$ with $|B(z)| > s$ for $|z| = \rho$. This implies that if k is sufficiently large, then $|B_k(z)| > s$ for $|z| = \rho$ and thus $\phi_k^{-1}(U_k) \subset D(0, \rho)$. Using Koebe's distortion theorem we find that

$$U_k \subset \phi_k(D(0, \rho)) \subset D\left(u_k, \frac{|\phi_k'(0)|\rho}{(1-\rho)^2}\right).$$

On the other hand, since $\operatorname{diam}(U_k) \geq \varepsilon$, we have $U_k \not\subset D\left(u_k, \frac{1}{4}\varepsilon\right)$. Thus $|\phi_k'(0)| \geq \varepsilon(1 - \rho)^2/4\rho$ and hence $|\phi'(0)| \geq \varepsilon(1 - \rho)^2/4\rho$. In particular, ϕ is not constant and is thus univalent. We deduce that $f^{n_k} = B_k \circ \phi_k^{-1} \to B \circ \phi^{-1}$ locally uniformly on $\phi(\mathbb{D})$. We shall now show that $\phi(\mathbb{D})$ contains a point of $J(f)$, which clearly contradicts the locally uniform convergence of f^{n_k} on $\phi(\mathbb{D})$. We choose $z_k \in U_k$ with $f^{n_k}(z_k) = 0$. Then $z_k \in J(f)$ because $0 = a \in J(f)$ by hypothesis. For large k we have $w_k := \phi_k^{-1}(z_k) \in D(0, \rho)$. Passing to a subsequence if necessary we may assume that $w_k \to w_0 \in \mathbb{D}$. Then $z_k = \phi_k(w_k) \to \phi(w_0) \in J(f) \cap \phi(\mathbb{D})$. ■

Theorem 2 follows easily from Theorem 1. However, it is also easily proved directly using the methods above. In fact, suppose that U is a component of $F(f)$ with $f^{n_k}|_U \to a$ as $k \to \infty$. Let r be as in the definition of semihyperbolicity. We may again assume that $r = 1$ and $a = 0$. We fix $c \in U$ and may assume that $f^{n_k}(c) \in \mathbb{D}$ for all k. We denote the component of $f^{-n_k}(\mathbb{D})$ that contains c by V_k and consider conformal maps $\phi_k : \mathbb{D} \to V_k$ with $\phi_k(0) = c$. Then $B_k := f^{n_k} \circ \phi_k$ is a Blaschke product of degree at most N, with $B_k(0) = f^{n_k}(c) \to 0$. As before we may assume that $\phi_k \to \phi$ with a univalent function ϕ and $B_k \to B$ with a Blaschke product B satisfying $B(0) = 0$. Thus $f^{n_k} \to B \circ \phi^{-1}$, contradicting the assumption that $f^{n_k} \to a = 0$ on U.

4. Proofs of Theorem 3 and Theorem 4

Proof of Theorem 3. Since ∂U is compact, we can choose the same $r > 0$ and $N \in \mathbb{N}$ in the hypothesis for all $a \in \partial U$. First we show that there exist θ with $0 < \theta < 1$ and a constant $c > 0$ such that for every $z \in \partial U$ and $n \in \mathbb{N}$ the Euclidean diameter of each component of $f^{-n}(D(z, \frac{1}{2}r))$ intersecting ∂U is less than $c\,\theta^n$. Our argument is similar to that of Yin [22]. We denote the module of the Grötzsch ring domain $\mathbb{D}\backslash\{x \mid 0 \leq x \leq r\}$ by $\mu(r)$. Grötzsch's theorem says that if A is an annulus separating 0 and $z \in \mathbb{D}$ from the unit circle, then the module $\operatorname{mod}(A)$ of A satisfies $\operatorname{mod}(A) \leq \mu(|z|)$. It follows that if A separates two points $z_1, z_2 \in \mathbb{D}$ from the unit circle, then $\operatorname{mod}(A) \leq \mu(|z_1 - z_2|/|1 - \overline{z_1}z_2|)$. This implies that if $a \in \mathbb{C}$, $R > 0$ and $C \subset D(a, \frac{1}{2}R)$ is compact and connected, then $\operatorname{mod}(D(a,R)\backslash C) \le \mu((4\operatorname{diam}_E(C)/5)R)$, where $\operatorname{diam}_E(\cdot)$ denotes the Euclidean diameter. We note that there exists $L \in \left(0,\frac{1}{2}\right)$ with $\mu(r) < \log(4/r)$ for all $r < L$; see [15, §II.2]. The argument used in the proof of Theorem 1 shows that there exists $n_0 \in \mathbb{N}$ such that if $n \ge n_0$, then $\operatorname{diam}_E(V) < \frac{1}{2}rL < \frac{1}{4}r$ for each component V of $f^{-n}(D(z,\frac{1}{2}r))$ intersecting ∂U. For simplicity we shall assume that $n_0 = 1$. We fix $n \in \mathbb{N}$ and $z_0 \in \partial U$, and consider a component V_n of $f^{-n}(D(z_0,\frac{1}{2}r))$ intersecting ∂U. We define $r_n := \operatorname{diam}_E(V_n)$. We take $z_n \in V_n \cap \partial U$ satisfying $f^n(z_n) = z_0$ and set $z_{n-k} = f^k(z_n)$ for $k = 1,2,\cdots,n-1$. Let V_k be a component of $f^{-k}(D(z_{n-k},\frac{1}{2}r))$ containing z_n. Then

$$f^k(V_{k+1}) \subset D\left(z_{n-k},\frac{r}{4}\right)$$

for $k = 0,1,\cdots,n-1$ and

$$V_0 := D\left(z_n,\frac{r}{2}\right) \supset V_1 \supset \cdots \supset V_n.$$

Since

$$\begin{aligned}
&\frac{\log 2}{2\pi}\\
&= \operatorname{mod}\left(D\left(z_n,\frac{r}{2}\right) \setminus \overline{D\left(z_n,\frac{r}{4}\right)}\right)\\
&\le \operatorname{mod}\left(D\left(z_n,\frac{r}{2}\right) \setminus \overline{V_1}\right)\\
&= \operatorname{mod}\left(f^k\left(V_k \setminus \overline{V_{k+1}}\right)\right)\\
&= \deg\left(f^k|_{V_k\setminus\overline{V_{k+1}}} : V_k \setminus \overline{V_{k+1}} \to f^k\left(V_k \setminus \overline{V_{k+1}}\right)\right) \cdot \operatorname{mod}\left(V_k \setminus \overline{V_{k+1}}\right)\\
&\le N \cdot \operatorname{mod}\left(V_k \setminus \overline{V_{k+1}}\right)
\end{aligned}$$

we have

$$\operatorname{mod}(V_k \setminus \overline{V_{k+1}}) \ge \frac{\log 2}{2\pi N},$$

for $k = 1,2,\cdots,n-1$. Since $V_1 \subset D(z_n,\frac{1}{4}r)$ the last inequality is also true for $k = 0$. Hence we have

$$\log\frac{5r}{2r_n} > \mu\left(\frac{4}{5}\cdot\frac{2}{r}r_n\right) \ge \operatorname{mod}(V_0 \setminus \overline{V_n}) \ge \sum_{k=0}^{n-1}\operatorname{mod}(V_k \setminus \overline{V_{k+1}}) \ge \frac{n\log 2}{2\pi N}.$$

We deduce that $c = 5r/2$ and $\theta = 2^{-1/2\pi N}$ have the required properties. The arguments used to show that an entire function cannot be semihyperbolic at a parabolic point or at a boundary point of a Siegel disk extend to the more general situation considered here where only branches of f^{-n} fixing ∂U are considered. Thus U is not a parabolic basin or a Siegel disk. Hence U is an attracting component and $f|_U$ is conjugate to a finite Blaschke product on the unit disk $\mathbb{D}$. Let q be the degree of that Blaschke product. In the rest of the argument we follow [4, §9.9], [7] and [21, §5.5]. As in [7, Theorems V.4.1 and VI.5.1] there exists $\rho \in (0,1)$, a compact, connected set $E \subset U$ and a homeomorphism $\varphi : \{z \in \mathbb{C} \mid \rho < |z| < 1\} \to U \setminus E$ such that $f(\varphi(z)) = \varphi(z^q)$ for $\rho^{1/q} < |z| < 1$. We may choose $t_0 \in (\rho,1)$ such that $\operatorname{dist}(\varphi(z),\partial U) < \frac{1}{2}r$ for $t_0 \le |z| < 1$. Here $\operatorname{dist}(\cdot,\cdot)$

denotes Euclidean distance. Then there exists $M \in \mathbb{N}$ and $t_1, \ldots, t_{M-1} \in (t_0, t_0^{1/q})$ with $t_0 < t_1 < \ldots < t_{M-1} < t_M := t_0^{1/q}$ such that for all $s \in [0, 2\pi]$ and for all $j \in \{1, \ldots, M\}$ there exists $z \in \partial U$ with $\{\varphi(te^{is}) \mid t_{j-1} \le t \le t_j\} \subset D(z, \frac{1}{2}r)$. We consider the sets

$$\Gamma(s,j,n) := \left\{ \varphi\left(te^{is}\right) \mid (t_{j-1})^{1/q^n} \le t \le (t_j)^{1/q^n} \right\}.$$

Thus for all s and j there exists $z \in \partial U$ with $\Gamma(s,j,n) \subset D(z, \frac{1}{2}r)$. Since $f^n(\varphi(te^{is})) = \varphi\left(t^{q^n} e^{iq^n s}\right)$ we have $f^n(\Gamma(s,j,n)) \subset \Gamma(q^n s, j, 0) \subset D(z, \frac{1}{2}r)$ for some $z \in \partial U$ so that $\operatorname{diam}_E(\Gamma(s,j,n)) \le c\,\theta^n$, for all s, j and n. We define $\gamma_n : [0, 2\pi] \to U \setminus E$, $\gamma_n(s) = \varphi(t_0^{1/q^n} e^{is})$. Then

$$\{\gamma_n(s), \gamma_{n+1}(s)\} \subset \Gamma(s,1,n) \cup \Gamma(s,M,n) \subset \bigcup_{j=1}^{M} \Gamma(s,j,n)$$

so that

$$|\gamma_n(s) - \gamma_{n+1}(s)| \le \operatorname{diam}_E\left(\bigcup_{j=1}^{M} \Gamma(s,j,n)\right) \le Mc\,\theta^n.$$

This implies that (γ_n) converges uniformly to a continuous function $\gamma : [0, 2\pi] \to \partial U$. Since f is entire we can deduce from the maximum principle that γ is actually a Jordan curve. ■

Proof of Theorem 4. To see that $J(f)$ is locally connected, due to a Theorem of Whyburn, it suffices to show that (i) the boundary of each Fatou component is locally connected and (ii) for an arbitrary $\varepsilon > 0$, the number of Fatou components whose diameter with respect to the spherical distance exceeds ε is finite. It follows from the hypotheses that each Fatou component is bounded. Furthermore, its boundary is a Jordan curve by Theorem 3. This implies the condition (i). Let $\{D(z_k, \frac{1}{2}r),\ z_k \in \partial U\}_{k=1}^{M}$ be an open covering of the boundary of an attracting component U, where r is as in the definition of semihyperbolicity. Since, by hypothesis, for every $n \in \mathbb{N}$ and every component V of $f^{-n}(U) \setminus \bigcup_{k=0}^{n-1} f^{-k}(U)$ we have $\deg(f^n|_V : V \to U) \le N$, we deduce that ∂V is covered by at most NM components of the preimages of the disks $D(z_k, \frac{1}{2}r)$ under f^n. Theorem 1 shows that $\operatorname{diam}(V)$ tends to 0 as n tends to infinity. This implies (ii). ■

5. Examples

The first example shows, as already mentioned in §2, that Mañé's characterization of semihyperbolicity for rational functions does not carry over to entire functions.

Example 1 *Let*

$$f(z) = \frac{z}{2} - \frac{1}{2\pi}\sin \pi z + c(\cos \pi z - 1),$$

where $c = 0.467763\cdots$ *is a solution of the equation* $\pi + 2\cos 2c\pi - 4c\pi \sin 2c\pi = 0$. *Then* f *has no asymptotic values, no parabolic periodic point and no recurrent critical point, but* f *is not semihyperbolic at* $1 \in J(f)$.

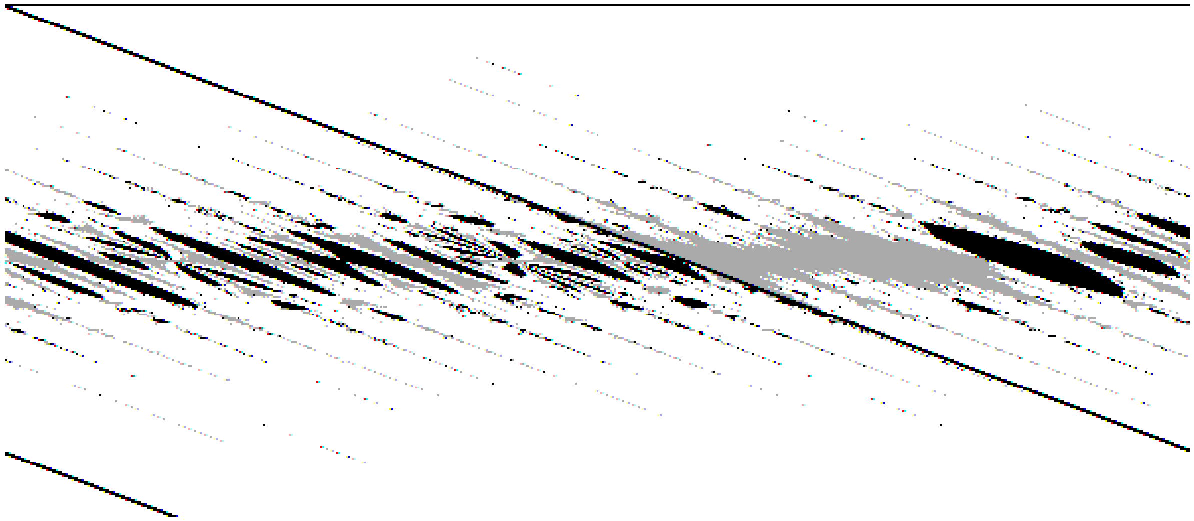

Figure 1. The Julia set of the function from Example 1. The basin of attraction of 0 is black, that of β is grey, and the Julia set is white. The range shown is $-4 \le \Re z \le 10$, $|\Im z| \le 3$. One can see that there are eight (resp. four) components of the basin of 0 with the point 8 (resp. 4) on their boundary.

Verification. Suppose that f has an asymptotic value, say $f(z) \to a$ as $z \to \infty$ on a path γ connecting 0 with ∞. Then $f(z)/z \to 0$ as $z \to \infty$ on γ. Also, $f(z)/z \to \frac{1}{2}$ as $z \to \infty$ along the positive or negative real axis. We may assume that γ does not intersect itself and intersects $\mathbb{R}$ only in 0. Thus we obtain four domains bounded by γ, $\overline{\gamma}$ and $\mathbb{R}$. In these domains $f(z)/z$ is unbounded. Hence $f(z)/z$ has at least four direct singularities over ∞. Since $f(z)/z$ has order 1, this contradicts the Denjoy-Carleman-Ahlfors theorem [20, §XI.4]. Thus f has no asymptotic values. Since

$$f'(z) = \frac{1}{2} - \frac{1}{2}\cos \pi z - c\pi \sin \pi z = \frac{1}{2} - \frac{\sqrt{1+4\pi^2c^2}}{2}\sin(\pi z + \theta),$$

where $\theta = 0.327959\cdots$ satisfies $\sin\theta = 1/\sqrt{1+4\pi^2c^2}$ and $\cos\theta = 2c\pi/\sqrt{1+4\pi^2c^2}$, the critical points of f are $z = 2n$ and $z = 2n+1-(2\theta/\pi) = 2n+1-0.208785\cdots$ for $n \in \mathbb{Z}$. Hence 0 is a superattracting fixed point. It is easy to see that $f(2k) = k$ for $k \in \mathbb{Z}$. Hence we have $f^n(2^n) = 1$ and the finite orbit $\{2^n, f(2^n) = 2^{n-1}, f^2(2^n) = 2^{n-2}, \cdots, f^{n-1}(2^n) = 2\}$ consists of n critical points. It follows that if $r > 0$ and U is the component of $f^{-n}(D(1,r))$ that contains 2^n, then $\deg(f^n|_U : U \to D(1,r)) \ge 2^n$. Hence f is not semihyperbolic at 1. The equation satisfied by c implies that $\alpha = f(1) = \frac{1}{2} - 2c = -0.435526\cdots$ is a fixed point of f. By calculation, we see that $f'(\alpha) = 1.838896\cdots$ so that α is repelling. We deduce that $\alpha \in J(f)$ and since $\alpha = f(1)$ this implies that $1 \in J(f)$.

Furthermore, we see that $\beta = f(-1) = \alpha - 1$ is a fixed point of f, with $f'(\beta) = 1 - f'(\alpha) = -0.838896\cdots$. Hence β is an attracting fixed point. The zeros of $f' - 1$ are given by $z = 2n+1$ and $z = 2n - (2\theta/\pi)$, with $n \in \mathbb{Z}$. By Rolle's theorem, any interval bounded by two (real) fixed points of f contains at least one zero of $f' - 1$.

Since there is exactly one zero of $f'-1$ in each of the intervals (β,α) and $(\alpha,0)$, these intervals contain no fixed points of f, and we have $f(x)>x$ for $x\in(\alpha,0)$ and $f(x)<x$ for $x\in(\beta,\alpha)$. Moreover, the interval $(\alpha,0)$ contains no critical point of f so that $f((\alpha,0))\subset(\alpha,0)$. Together with $f(x)>x$ for $x\in(\alpha,0)$ this implies that $(\alpha,0]$ is contained in the attracting basin of 0. Next we note that if $x<\beta$, then $f(x)>x$. This follows from the fact that the local minima of $f(x)-x$ are taken at the points $2n+1$ and that $f(2n+1)=n+\alpha>2n+1$ for $n\le -2$. A similar argument shows that $f(x)<x$ for $x>0$. We now show that $(-\infty,\alpha)$ is contained in the attracting basin of β. To do so we note that $\xi=-1.208785\cdots$ is the only critical point of f contained in the interval $(-2,0)$. We have $f(\xi)=-1.539919\cdots$ and $f^2(\xi)=-1.337119\cdots$. Since there is no critical point in the interval $(f(\xi),\xi)$, we obtain $f([f(\xi),\xi])=[f(\xi),f^2(\xi)]\subset[f(\xi),\xi]$. If we denote by I_n the closed interval bounded by $f^{n-1}(\xi)$ and $f^n(\xi)$, the last equation takes the form $f(I_1)=I_2\subset I_1$, and induction shows that $f(I_n)=I_{n+1}\subset I_n$ for all $n\in\mathbb{N}$. It is not difficult to see that there exists $\eta<1$ such that $|f'(x)|\le\eta$ for $x\in I_3$. It follows that $f^n(x)\to\beta$ as $n\to\infty$, uniformly for $x\in I_3$, and hence for $x\in I_1$. We also have $f([\xi,-1])=[f(\xi),f(-1)]=[f(\xi),\beta]\subset I_1$, and thus $f^n(x)\to\beta$ as $n\to\infty$ also for $x\in[\xi,-1]$. For $x<\beta$ we have $x<f(x)\le f(-2)=-1$. This implies that $f^n(x)\to\beta$ as $n\to\infty$ for $x<\beta$. Using $f(x)<x$ for $\beta<x<\alpha$ we then obtain $f^n(x)\to\beta$ as $n\to\infty$ for all $x\in(-\infty,\alpha)$. Finally, using $f(x)<x$ for $x>0$ we see that for every real number its orbit converges to 0 or β, or it is eventually mapped onto α. In particular, this is the case for every critical point. Thus f has no parabolic periodic point and no recurrent critical point. ■

In the following examples we use Corollaries 2 and 3 to rule out wandering domains. In Example 2 we can then use Theorem 4 to prove that $J(f)$ is locally connected, while $J(f)=\mathbb{C}$ in Example 3.

Example 2 *Let*

$$f(z)=\frac{az}{\pi^2-4z}\cos\sqrt{z}.$$

There exists A such that if $\pi^2<a<A$, then f has the following properties: f is semihyperbolic and has an attracting fixed point such that $F(f)$ consists of its basin. Also, $J(f)$ contains infinitely many critical values. Furthermore, $J(f)$ is locally connected.

Verification. Similarly as in Example 1 we see that there exists no asymptotic value. Next we note that all critical points of f are real. For example, this follows from the fact that polynomials with only real zeros have only real critical points, and f is a locally uniform limit of such polynomials. This argument also shows that between two zeros of f there is exactly one critical point. The critical points are thus given by $c_1^+<c_1^-<c_2^+<\cdots$ with $c_1^+\in(0,(\frac{3}{2}\pi)^2)$, $c_n^+\in((2n-\frac{3}{2})^2\pi^2,(2n-\frac{1}{2})^2\pi^2)$ for $n\ge 2$ and $c_n^-\in((2n-\frac{1}{2})^2\pi^2,(2n+\frac{1}{2})^2\pi^2)$ for $n\in\mathbb{N}$. Clearly we have $f(c_n^+)>0$ and $f(c_n^-)<0$. Now the maximum of f in the interval $((2n-\frac{3}{2})^2\pi^2,(2n-\frac{1}{2})^2\pi^2)$ is attained at c_n^+ for

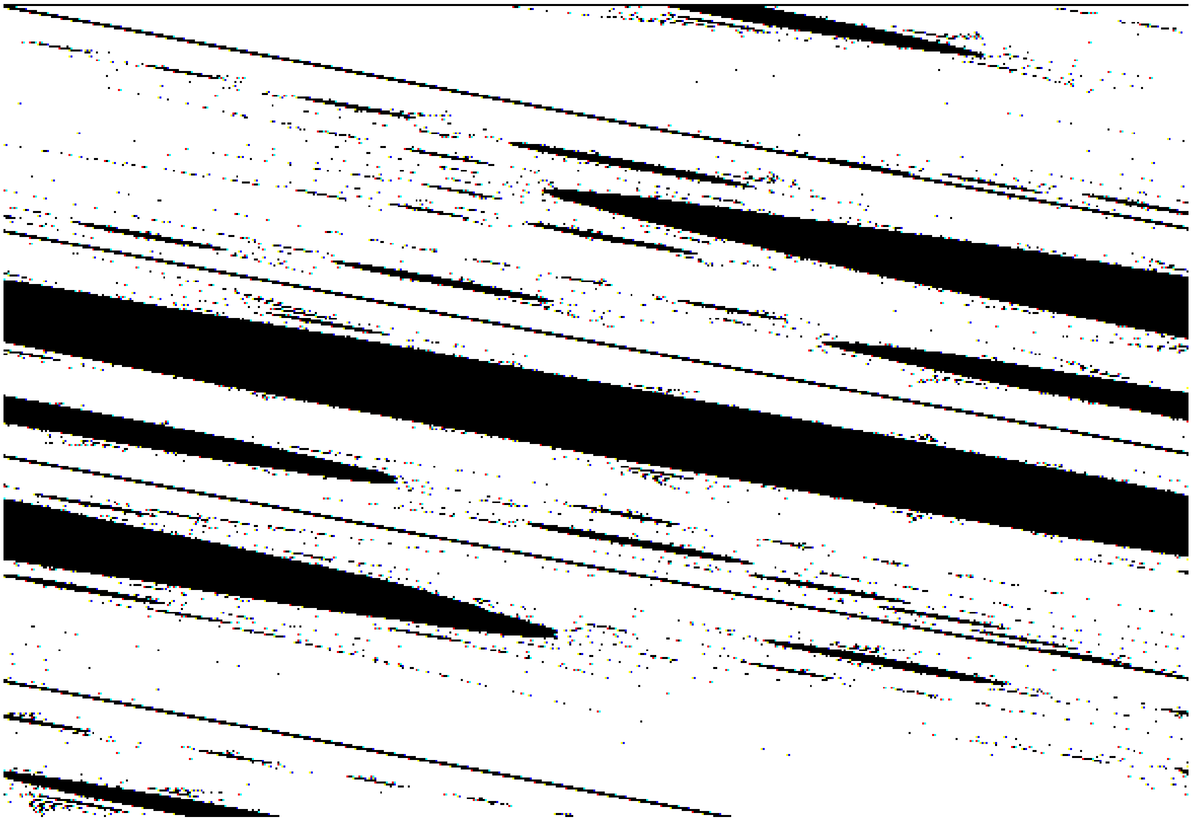

Figure 2. The Julia set of the function from Example 2 for $a = 3\pi^2$. In this case $\alpha = \pi^2$ is an attracting fixed point. The basin of attraction of α is black and the Julia set is white. The range shown is $-25 \le \Re z \le 150$, $|\Im z| \le 60$.

$n \ge 2$, and it tends to $a/4$ as $n \to \infty$. Hence $\lim_{n\to\infty} f(c_n^+) = a/4$. Similarly, we have $\lim_{n\to\infty} f(c_n^-) = -a/4$. Hence f belongs to $\mathcal{B}$.

We see that f is a unimodal map on the interval $[0, 9\pi^2/4]$. Furthermore we have $f(0) = f(9\pi^2/4) = 0$ and $f'(0) = a/\pi^2 > 1$ for $a > \pi^2$ so that 0 is a repelling fixed point. The zeros of f'' are located in the intervals bounded by the zeros of f'. A numerical computation shows that the smallest zero of f'' is at $z = 18.76159\cdots$. We find that f is concave in the interval $[0, 18.76159\cdots]$. Since f attains its maximum in this interval at $c_1^+ = 8.07147\cdots$ we conclude that f has exactly one fixed point $\alpha = \alpha(a)$ in $(0, 9\pi^2/4)$. Moreover, $\alpha(a) \to 0$ as $a \to \pi^2$ and $\alpha(a) \to 9\pi^2/4$ as $a \to \infty$, and α is an increasing function of a. We also see that $f'(\alpha) \in [0,1)$ for $\alpha \in (0, c_1^+]$ and that $f'(\alpha)$ is a decreasing function of a as long as $\alpha \in [c_1^+, 18.76159\cdots]$. A numerical computation shows that $a = 43.90495\cdots$ and $\alpha = 12.5642\cdots$ are a solution of the simultaneous equations $f(\alpha) = \alpha$ and $f'(\alpha) = -1$. We set $A = 43.90495\cdots$ and conclude that if $\pi^2 < a < A$, then $1 > f'(\alpha) > -1$ so that α is an attracting fixed point.

Since $\pi^2 \cosh y - (\pi^2 + 4y^2) > 0$ for $y > 0$ and $a > \pi^2$, we have $f(x) < x$ for $x < 0$. Hence $f^n(x) \to -\infty$ as $n \to \infty$ for $x < 0$ and thus $(-\infty, 0)$ is contained in $J(f)$ because $f \in \mathcal{B}$. Thus $c_n^- \in J(f)$ for all $n \in \mathbb{N}$. We denote by U the immediate attractive basin of α. Then U contains at least one critical point. Since U is simply connected and symmetric with respect to the real axis and since $c_n^- \in J(f)$ for all $n \in \mathbb{N}$, we conclude that $c_n^+ \notin U$ for $n \geq 2$. Thus $c_1^+ \in U$.

We shall show now that in fact $(0, c_1^+] \subset U$. Let I be the closed interval with endpoints α and c_1^+. Using the simple connectivity and symmetry of U as before we see that $I \subset U$. If $\alpha \leq c_1^+$ so that $I = [\alpha, c_1^+]$ we have $x < f(x) < \alpha$ for $0 < x < \alpha$ and this implies that $(0, \alpha) \in U$. Suppose now that $c_1^+ < \alpha$ so that $I = [c_1^+, \alpha]$. Then $f(I) = [\alpha, f(c_1^+)] \subset U$. For $0 < x < \alpha$ we have $x < f(x) \leq f(c_1^+)$, and this implies that $(0, \alpha) \in U$. Altogether we find that $(0, c_1^+] \subset (0, \alpha) \cup I \subset U$ and hence that $(0, f(c_1^+)] \subset U$ in any case.

Next we show that $f(c_1^+) > f(c_n^+)$ for $n \geq 2$. To do so, we note that if $n \geq 2$, then $c_n^+ \geq (2n - \frac{3}{2})^2\pi^2 \geq (\frac{5}{2}\pi)^2$ and thus

$$f(c_n^+) \leq \frac{ac_n^+}{4c_n^+ - \pi^2} = \frac{a}{4 - \pi^2/c_n^+} \leq \frac{a}{4 - \pi^2/(\frac{5}{2}\pi)^2} = \frac{25a}{96}.$$

On the other hand, a numerical computation yields $c_1^+ = 8.071473\cdots$ and $f(c_1^+) = 0.343930\cdots a > 25a/96$. Hence the $f(c_n^+)$ are contained in U. Using the simple connectivity and symmetry as above we see that the Fatou components containing the c_n^+ are mutually disjoint. We conclude that $\deg(f^n|_V : V \to U) \leq 2$ for $n \in \mathbb{N}$ and every component $V \neq U$ of $f^{-n}(U) \setminus \bigcup_{k=0}^{n-1} f^{-k}(U)$.

A similar consideration shows that if $x \in \mathbb{R} \cap J(f)$, then there exists $r > 0$ such that $\deg(f^n|_V : V \to D(x,r)) \leq 2$ for $n \in \mathbb{N}$ and every component V of $f^{-n}(D(x,r))$. And of course, if $z \in \mathbb{C} \backslash \mathbb{R}$ and $r < |\Im z|$, then f is univalent on every component $f^{-n}(D(x,r))$ for $n \in \mathbb{N}$. From the above, we see that f is semihyperbolic. It follows from Corollary 3 that $F(f)$ is equal to the attracting basin of α.

There exists a component L of $f^{-1}((-\infty, f(c_1^-)])$ which intersects $\mathbb{R}$ only in c_1^-. Clearly U does not intersect L. Actually L is a curve which has an asymptotic curve of the form $x = by^2 + c$, where $z = x + iy$ and $b < 0$. This follows since the function $z \mapsto \cos\sqrt{z}$ is a composed function of $z \mapsto \sqrt{z}$ and $z \mapsto \cos z$ and has no critical point in the negative real axis. Furthermore, $az/(\pi^2 - 4z)$ tends to the negative constant $-a/4$ as $|z| \to \infty$. We denote the domain bounded by L which contains U by D. For sufficiently large R, we see $|f(z)| \approx \exp|\Im\sqrt{z}|$ for $z \in D$ and $|z| > R$ and hence we have $|f(z)| > |z|$ for $z \in D$, $|z| > R$. This implies that $U \subset D \cap D(0,R)$ so that U is bounded. From Theorem 4, we see that $J(f)$ is locally connected. ■

Example 3 *Let*

$$f(z) = \frac{z}{\pi^2 - 4z}\cos\sqrt{z} - b,$$

with b chosen so large that $f(x) < x$ for all $x \in \mathbb{R}$. Then f is semihyperbolic and $J(f) = \mathbb{C}$. Moreover, f has infinitely many critical values.

Verification. We note that $f' - 1$ has only one real zero $c = -18.5261\cdots$. It suffices to choose b such that $f(c) < c$. This shows that $b > 10.36071\cdots$ satisfies the hypothesis.

As before we see that there are no asymptotic values and that all critical points are in the positive real axis. Hence f belongs to $\mathcal{B}$. By the assumption $f(x) < x$ for all $x \in \mathbb{R}$, the orbit of each critical point tends to infinity. Thus $F(f)$ has no periodic component. We see that f is semihyperbolic by the argument before. Hence f has no wandering domain where limit functions are finite by Corollary 2. Since $f \in \mathcal{B}$, we conclude $J(f) = \mathbb{C}$. ■

Our last example shows that semihyperbolicity at the boundaries of Fatou components may suffice to prove the local connectivity of a Julia set.

Example 4 *Let*

$$f(z) = a - (a+\pi)\frac{\sin z}{z}$$

with $a = 3.0008\cdots$ *which satisfies* $f(a) = f^5(a)$*. Then* f *is semihyperbolic at every point of* $J(f)$ *except* a, $f(a)$, $f^2(a)$, $f^3(a)$ *and* $f^4(a)$*, and* f *has no wandering domain. Moreover* $J(f)$ *is locally connected.*

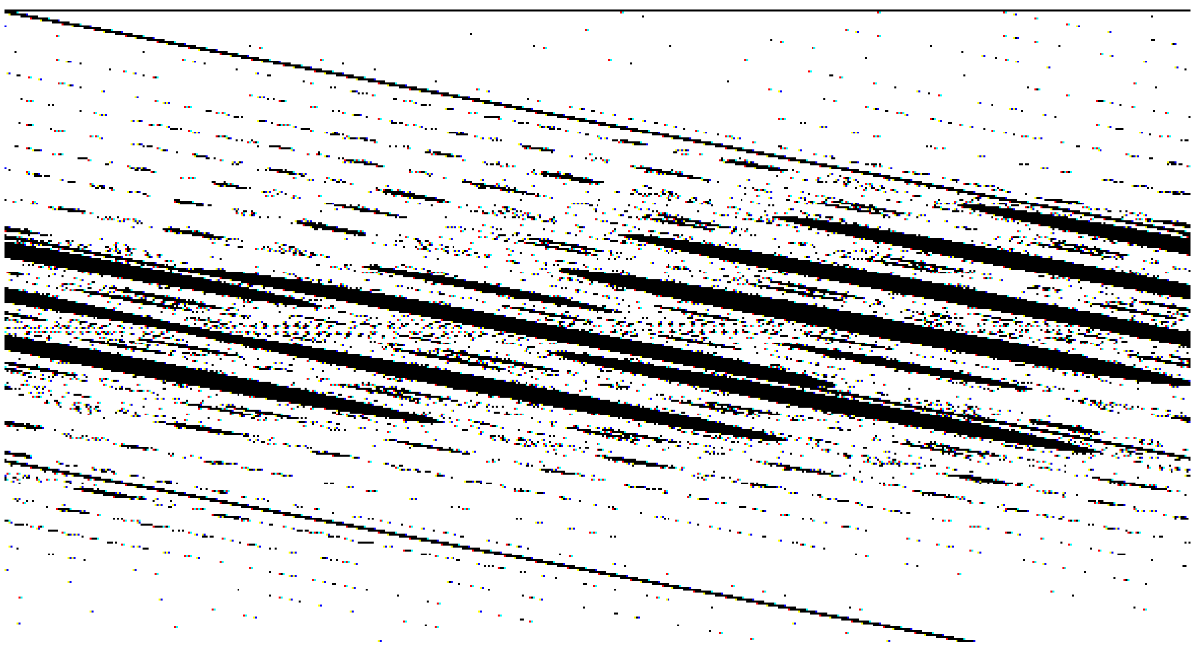

Figure 3. The Julia set of the function from Example 4. The basin of attraction of α is black and the Julia set is white. The range shown is $-5 \le \Re z \le 25$, $|\Im z| \le 8$. One can see that $c_1^+ = 4.493409\cdots \in U$, that $c_2^+ = 10.90412\cdots$ and $c_3^+ = 17.22075\cdots$ are in preimages of U and that $c_4^+ = 23.51945\cdots \in J(f)$.

Verification. It is clear that a is an asymptotic value and, by the assumption, a is preperiodic and $f(a)$ is a periodic point with period 4. A numerical computation shows that the periodic cycle $\{f(a), f^2(a), f^3(a), f^4(a)\}$ is repelling. As in Example 2 we see that all critical points are in the real axis and thus $P(f)$ is contained in the real axis. We denote the critical points by 0, $\pm c_i^+$ and $\pm c_i^-$ for $i \in \mathbb{N}$, with $0 < \pi < c_1^+ < c_1^- < c_2^+ < \cdots$, and they satisfy $\lim_{i\to\infty} \pm c_i^+ = \lim_{i\to\infty} \pm c_i^- = \pm\infty$, $f(\pm c_i^+) > a$, $0 < f(\pm c_i^-) < a$, $\lim_{i\to\infty} f(\pm c_i^+) = a$ and $\lim_{i\to\infty} f(\pm c_i^-) = a$.

Numerical computation shows that f has an attracting fixed point $\alpha = 4.31283\cdots \in [\pi, 2\pi]$ and a repelling fixed point $\beta = 3.30750\cdots$. Similarly as in Example 2 we see that the interval $(\beta, \alpha]$ is contained in the immediate basin U of α. Moreover, we have $c_1^+ \in U$.

Next we note that 0 is the only critical point of f contained in the interval $[-\pi, \pi]$ and that $f(0) = -\pi$ and $f(\pm\pi) = a$. Thus f restricted to $[-\pi, \pi]$ is a unimodal map satisfying $f([-\pi, \pi]) \subset [-\pi, \pi]$. A periodic component of $F(f)$ intersecting $[-\pi, \pi]$ is an attracting or parabolic basin and thus contains a critical point of f. On the other hand, it is also simply connected. Arguing as in Example 2 we see that there is no periodic component of $F(f)$ intersecting $[-\pi, \pi]$.

For $x \in [\pi, \beta)$ we have $f(x) < x$ and thus $f^n(x) \in [-\pi, \pi]$ for sufficiently large n. A numerical computation yields $c_2^+ = 10.90412\cdots$, $c_3^+ = 17.22075\cdots$, $f(c_2^+) = 3.56175\cdots$ and $f(c_3^+) = 3.35688\cdots$. This implies that $\{f(c_2^+), f(c_3^+)\} \subset (\beta, \alpha)$ so that $\lim_{n\to\infty} f^n(\pm c_i^+) = \alpha$ for $i \in \{2, 3\}$. Note that the Fatou component containing c_2^+ and the Fatou component containing c_3^+ are different because $c_2^+ < 4\pi < c_3^+$ and $f(4\pi) = a \in J(f)$. Since $c_i^+ > 13\pi/2$ for $i \geq 4$, we have

$$f(c_i^+) = a - (a + \pi)\frac{\sin c_i^+}{c_i^+} \leq a + \frac{2(a+\pi)}{13\pi} = 3.3016\cdots < \beta$$

by numerical computation. Hence $f^n(c_i^+) \in [-\pi, \pi] \subset J(f)$ for sufficiently large n if $i \geq 4$. Also, it is not difficult to see that $f(\pm c_i^-) \in [-\pi, \pi]$ for all $i \in \mathbb{N}$. By an argument similar to that in the verification of Example 2 we now find that f is semihyperbolic at every point of $J(f)$ except a, $f(a)$, $f^2(a)$, $f^3(a)$ and $f^4(a)$.

By Theorem 2, if f has a wandering domain, then the only possible finite limit functions there are a, $f(a)$, $f^2(a)$, $f^3(a)$ and $f^4(a)$. Since $f(a)$, $f^2(a)$, $f^3(a)$ and $f^4(a)$ form a repelling periodic cycle, there exists no wandering domain with finite limit functions. Hence f has no wandering domain and no Baker domain because f belongs to $\mathcal{B}$. Thus every Fatou component is eventually mapped on the attracting component U containing α. It is clear that $\deg(f^n|_V\colon V \to U) \leq 2$ for $n \in \mathbb{N}$ and every component $V \neq U$ of $f^{-n}(U) \setminus \bigcup_{k=0}^{n-1} f^{-k}(U)$.

The imaginary axis is mapped into the negative real axis. There exist a curve L containing c_1^- which is mapped into $(-\infty, \pi)$. It has an asymptotic line $x = 2\pi$. We denote the domain bounded by the imaginary axis and L by D. Since U is a simply connected invariant component, we have $U \subset D$. For sufficiently large M, we have $f(z) \approx e^{-iz}$ in $D \cap \{z \mid \Im z > M\}$ and $f(z) \approx e^{iz}$ in $D \cap \{z \mid \Im z < -M\}$. Thus $|f(z)| > |z|$ for $z \in D$, $|z| > M$, if M is sufficiently large. Hence U is bounded. Because U is symmetric with respect to real axis, $f(a)$, $f^2(a)$, $f^3(a)$ and $f^4(a)$ are not on ∂U and neither is a. From Theorem 3, ∂U is a Jordan curve.

A point which is not on the boundary of any Fatou component is called a buried point. Since every Fatou component is eventually mapped to U and since a, $f(a)$, $f^2(a)$,

$f^3(a)$ and $f^4(a)$ are not on ∂U, we conclude that a, $f(a)$, $f^2(a)$, $f^3(a)$ and $f^4(a)$ are buried points. By Theorem 4 we see that $J(f)$ is locally connected. ■

Finally we briefly mention the example

$$f(z) = \pi^2 - a\frac{\sin\sqrt{z}}{\sqrt{z}}$$

considered in [6]. It was shown there that $f \in \mathcal{B}$ and that $P(f) \cap J(f)$ has no finite limit points if $\pi^2 < a < 2\pi^2$. Then the main result of [6] was used to show that f has no wandering domains for a in this range. The methods of this paper allow to show that f has no wandering domains for $\pi^2 < a < A = 91.1046\cdots$, where A is chosen such that the smallest positive fixed point w of f satisfies $f'(w) = -1$ if $a = A$. Moreover, $J(f) \cap P(f)$ and in fact $J(f) \cap \operatorname{sing}(f^{-1})$ have finite limit points for $2\pi^2 \leq a < A$, so that the method of [6] does not seem to be applicable. The arguments used to show this are similar to the ones used before. We omit the details.

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